

BRUNDAN-KAZHDAN-LUSZTIG CONJECTURE FOR GENERAL LINEAR LIE SUPERALGEBRAS

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ABSTRACT. In the framework of canonical and dual canonical bases of Fock spaces, Brundan in 2003 formulated a Kazhdan-Lusztig-type conjecture for the characters of the irreducible and tilting modules in the BGG category for the general linear Lie superalgebra for the first time. In this paper, we prove Brundan's conjecture and its variants associated to all Borel subalgebras in full generality.

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1. INTRODUCTION

1.1. Background. In the classical papers [Kac1, Kac2], Kac in 1970's initiated the study of representations of Lie superalgebras, including the general linear Lie superalgebra $\mathfrak{gl}(m|n)$ and other basic Lie superalgebras. Realizing the difficulty of generalizing the Weyl character formula for finite-dimensional irreducible modules to the Lie superalgebra setting, Kac found a Weyl type character formula for a class of so-called *typical* finite-dimensional simple modules. Since then, there have been numerous attempts to obtain further results toward the irreducible character problem for Lie superalgebras (see the bibliography of a forthcoming book [CW2] for a partial list).

Around 1996, Serganova [Se] developed a mixed algebraic and geometric approach to provide an algorithm for obtaining the finite-dimensional irreducible characters for $\mathfrak{gl}(m|n)$. For lack of a general conceptual framework, there was virtually no serious attempt in addressing the irreducible character problem in a Bernstein-Gelfand-Gelfand (BGG) category \mathcal{O} for Lie superalgebras such as $\mathfrak{gl}(m|n)$, until the work of Brundan.

1.2. BKL conjecture. In his 2003 seminal paper [Br1] Brundan formulated an elegant conjecture on the characters for the irreducible modules and tilting modules in the full BGG category $\mathcal{O}^{m|n}$ of $\mathfrak{gl}(m|n)$ -modules for the first time (also see [Br3] for a similar formulation for the Lie superalgebra $\mathfrak{q}(n)$). Brundan's formulation was in terms of canonical and dual canonical bases of Lusztig and Kashiwara [Lu1, Ka] in a Fock space $\mathbb{T}^{\mathbf{b}_{\text{st}}} := \mathbb{V}^{\otimes m} \otimes \mathbb{W}^{\otimes n}$ (or rather in some suitable completion $\widehat{\mathbb{T}}^{\mathbf{b}_{\text{st}}}$ of $\mathbb{T}^{\mathbf{b}_{\text{st}}}$), where \mathbb{V} is the natural module of the quantum group $U_q(\mathfrak{gl}_{\infty})$ and \mathbb{W} is the restricted dual of \mathbb{V} . In this formulation, the Verma modules correspond to the standard monomial basis, the irreducible modules to the dual canonical basis, and the tilting modules to the canonical basis in $\widehat{\mathbb{T}}^{\mathbf{b}_{\text{st}}}$.

In other words, the entries of the transition matrices between the standard monomial and (dual) canonical bases define the Brundan-Kazhdan-Lusztig (BKL) polynomials, whose values at $q = 1$ solve the multiplicity problem of irreducible and tilting characters when expressed in terms of Verma characters. In a nutshell, the category $\mathcal{O}^{m|n}$ categorifies the Fock space $\mathbb{T}^{\mathbf{b}_{\text{st}}}$ and its (dual) canonical bases. (In the Introduction below, we will ignore completely the issues of completions of various Fock spaces, though it will take some considerable portion of this paper to take care of such issues properly.)

Brundan's conjecture can be adapted for various parabolic categories of $\mathfrak{gl}(m|n)$ -modules, where an even Levi subalgebra of $\mathfrak{gl}(m|n)$ in the block matrix form of size $(m_1, m_2, \dots, m_{\ell} | n_1, \dots, n_r)$ gives rise to a Fock space in terms of q -wedge subspaces as follows:

$$\wedge^{m_1} \mathbb{V} \otimes \dots \otimes \wedge^{m_{\ell}} \mathbb{V} \otimes \wedge^{n_1} \mathbb{W} \otimes \dots \otimes \wedge^{n_r} \mathbb{W}.$$

The validity of Brundan's conjecture on the full BGG category implies the validity of all such parabolic versions. In the same paper Brundan [Br1] proved a maximal parabolic version of his general conjecture, which can be phrased that the category $\mathcal{F}^{m|n}$ of finite-dimensional $\mathfrak{gl}(m|n)$ -modules categorifies $\wedge^m \mathbb{V} \otimes \wedge^n \mathbb{W}$ and its (dual) canonical bases.

In the case when either m or n is zero, Brundan's formulation reduces to the by now well-known reformulation of the classical Kazhdan-Lusztig conjecture [KL1, KL2] (and a tilting module version [So1, So2]) for category \mathcal{O} of general linear Lie algebras, which takes advantage of the Schur-Jimbo duality [Jim]. The formulation of Brundan's conjecture in terms of quantum groups and canonical basis is a conceptual way of getting around the well-known difficulty that the Weyl group $W = \mathfrak{S}_m \times \mathfrak{S}_n$ and its associated Hecke algebra are insufficient to control the linkage principle for $\mathfrak{gl}(m|n)$.

As is well known, there exists simple systems for $\mathfrak{gl}(m|n)$ which are not conjugate under the action of the Weyl group W , for $m, n \geq 1$. The W -conjugacy classes of simple systems are in bijection with what we call $0^m 1^n$ -sequences \mathbf{b} (there are $\binom{m+n}{n}$ of them in total). To each such \mathbf{b} is associated a Borel subalgebra \mathfrak{b} of $\mathfrak{gl}(m|n)$, and these Borel subalgebras are not conjugate to each other. It has been expected (cf. Kujawa's thesis [Ku]) that Brundan's conjecture affords variants in terms of Fock spaces $\mathbb{T}^{\mathbf{b}}$, which is a q -tensor space with m tensor factors isomorphic to \mathbb{V} and n factors isomorphic to \mathbb{W} , determined by the sequence \mathbf{b} (see (2.1) for a precise definition), for each \mathbf{b} . We will refer to all these \mathbf{b} -variants as *Brundan-Kazhdan-Lusztig (BKL) conjecture*. Kujawa's work [Ku] provided a first supporting evidence on the crystal basis level for the BKL conjecture. In this paper, we shall need and hence formulate such \mathbf{b} -variants of

Brundan's conjecture precisely, which requires some suitable completions of the corresponding Fock space $\mathbb{T}^{\mathbf{b}}$ in order to construct the (dual) canonical bases. The original Brundan's conjecture is associated to the standard sequence $\mathbf{b}_{\text{st}} = (0, \dots, 0, 1, \dots, 1)$.

It is clear from the beginning that Brundan's conjecture is a central problem in representation theory of Lie superalgebras. As the proof of the classical KL conjecture [KL1, KL2] was completed in [BB, BK] independently using deep geometric machinery and the BKL conjecture includes the type A KL conjecture as a special case, there seemed to be little hope of proving Brundan's conjecture directly due to the inadequate development of the geometric approach in super representation theory. However, Brundan's work convinced the authors that a general and conceptual approach, though likely completely novel, to the representation theory of Lie superalgebras might still be possible.

1.3. Goal. The goal of this paper is to prove the BKL conjecture for the BGG category $\mathcal{O}^{m|n}$ in full generality. The proof consists of two major steps. First, via an extension of the super duality approach developed earlier by the authors, we establish an inductive procedure on n of proving the BKL conjecture for $\mathfrak{gl}(m|n+1)$ based on the validity of *one* \mathbf{b} -variant of the BKL conjecture for $\mathfrak{gl}(m|n)$, for every m . Our second main step is to show that the \mathbf{b} -variants of BKL conjecture for all $0^m 1^n$ -sequences \mathbf{b} are equivalent to each other.

1.4. Super duality. Let us explain the super duality in some detail, which was already used to solve some distinguished parabolic versions of the BKL conjecture.

A precise connection, which was christened as Super Duality, between representation theory of $\mathfrak{gl}(m|n)$ and that of $\mathfrak{gl}(m+n)$ was formulated in [CWZ] and in full generality in [CW1] for the first time. Super duality is a (conjectural at that time) category equivalence between suitable parabolic module categories of $\mathfrak{gl}(m|n)$ and $\mathfrak{gl}(m+n)$ at the $n \mapsto \infty$ limit. Such a connection in the most special case [CWZ] was supported by, and in turn implies Brundan's solution for the irreducible and tilting characters in the category $\mathcal{F}^{m|n}$. The conjectured general super duality was shown in [CW1] to imply the distinguished parabolic versions of BKL conjecture whose corresponding Fock spaces are of the form $\wedge^{m_1} \mathbb{V} \otimes \dots \otimes \wedge^{m_\ell} \mathbb{V} \otimes \wedge^n \mathbb{W}$. One bonus consequence of the super duality approach is that the BKL polynomials in these distinguished parabolic categories are shown to be exactly the classical parabolic Kazhdan-Lusztig polynomials.

In [CL], a powerful yet elementary approach was developed to prove the super duality conjecture of [CW1]. One notable feature of [CL] is that it does not rely on the results of [Br1] a priori, and hence the parabolic versions of the BKL conjecture as formulated in [CW1] followed. There is yet another independent and complete solution by Brundan and Stroppel [BS] toward the irreducible and tilting character problem for the maximal parabolic category $\mathcal{F}^{m|n}$. As far as the full BGG category $\mathcal{O}^{m|n}$ is concerned, almost nothing is known so far. The super duality approach has been further developed in [CLW] for the ortho-symplectic Lie superalgebras, where the irreducible and tilting characters were shown to be expressible in terms of Verma characters via classical Kazhdan-Lusztig polynomials.

1.5. Strategy of proof. Now we explain in more detail the outline of our proof of the BKL conjecture. The BKL conjecture for the BGG category $\mathcal{O}^{m|n}$ with respect to the standard Borel subalgebra will be abbreviated as $\text{BKL}(m|n)$. Let $\mathfrak{h}_{m|n}$ be the Cartan subalgebra of diagonal matrices in $\mathfrak{gl}(m|n)$. We shall denote by $\text{BKL}(m|n + \underline{k})$ the (parabolic) Brundan conjecture for the parabolic category of $\mathfrak{gl}(m|n + k)$ -modules with Levi subalgebra $\mathfrak{h}_{m|n} \oplus \mathfrak{gl}(k)$, for $k \leq \infty$. We then let $\text{BKL}(m|n|\underline{k})$ denote the BKL conjecture for the parabolic BGG category of $\mathfrak{gl}(m + k|n)$ -modules associated to the $0^{m+k}1^n$ -sequence $(0^m, 1^n, 0^k)$ and with Levi subalgebra $\mathfrak{h}_{m|n} \oplus \mathfrak{gl}(k)$, for $k \leq \infty$.

The overall strategy for establishing the BKL conjecture is by induction on n . The inductive procedure, denoted by $\text{BKL}(m|n) \forall m \implies \text{BKL}(m|n + 1)$, is divided into the following steps:

- (1.1) $\text{BKL}(m + k|n) \forall k \implies \text{BKL}(m|n|k) \forall k$, by changing Borels
- (1.2) $\implies \text{BKL}(m|n|\underline{k}) \forall k$, by passing to parabolic
- (1.3) $\implies \text{BKL}(m|n|\underline{\infty})$, by taking $k \mapsto \infty$
- (1.4) $\implies \text{BKL}(m|n + \underline{\infty})$, by super duality
- (1.5) $\implies \text{BKL}(m|n + 1) \forall m$, by truncation.

It is instructive to write down the Fock spaces corresponding to the steps above:

$$\begin{aligned}
 \mathbb{V}^{\otimes(m+k)} \otimes \mathbb{W}^{\otimes n} \quad \forall k &\implies \mathbb{V}^{\otimes m} \otimes \mathbb{W}^{\otimes n} \otimes \mathbb{V}^{\otimes k} \quad \forall k \\
 &\implies \mathbb{V}^{\otimes m} \otimes \mathbb{W}^{\otimes n} \otimes \wedge^k \mathbb{V} \quad \forall k \\
 &\implies \mathbb{V}^{\otimes m} \otimes \mathbb{W}^{\otimes n} \otimes \wedge^{\infty} \mathbb{V} \\
 &\implies \mathbb{V}^{\otimes m} \otimes \mathbb{W}^{\otimes n} \otimes \wedge^{\infty} \mathbb{W} \\
 &\implies \mathbb{V}^{\otimes m} \otimes \mathbb{W}^{\otimes(n+1)} \quad \forall m.
 \end{aligned}$$

While our proof is purely algebraic, it is ultimately based on the geometric proof of the original KL conjecture. The base case for the induction, $\text{BKL}(m|0)$, is equivalent to the original Kazhdan-Lusztig conjecture [KL1] for $\mathfrak{gl}(m)$, which is a theorem of [BB] and [BK] (also see [BGS, Vo]; the tilting module characters were due to [So1, So2]). Step (1.2) can be regarded as a generalization of Deodhar [Deo], Soergel [So1] and Brundan [Br1]. Steps (1.3)–(1.5) are generalizations of our earlier work in [CWZ, CW1, CL], in which we establish the compatibilities between various constructions on the categorical level and their counterparts on the Fock space level.

Step (1.1) is a special case of a key new result of this paper, which states that all \mathbf{b} -variants of the BKL conjecture (for fixed m, n) are equivalent. We are reduced to compare the (dual) canonical bases of the Fock spaces $\mathbb{T}^{\mathbf{b}}$ and $\mathbb{T}^{\mathbf{b}'}$ associated to adjacent $0^m 1^n$ -sequences \mathbf{b} and \mathbf{b}' . Here being adjacent corresponds to differing by an odd reflection on the corresponding Borel subalgebras. To that end, *parabolic monomial bases* for $\mathbb{T}^{\mathbf{b}}$ and $\mathbb{T}^{\mathbf{b}'}$ are introduced, and they are used to match the (dual) canonical bases in $\mathbb{T}^{\mathbf{b}}$ and $\mathbb{T}^{\mathbf{b}'}$. Note that parabolic monomial bases admit counterparts in the category $\mathcal{O}^{m|n}$. We also show how tilting modules transform under odd reflections, and

establish a remarkable property that every tilting module in $\mathcal{O}^{m|n}$ has \mathbf{b} -Verma flags for all $0^m 1^n$ -sequences \mathbf{b} . Now (1.1) follows from such constructions and results.

1.6. Applications. We show in Proposition 6.4 that the categories $\mathcal{O}_{\mathbf{b}}^{m|n}$ are identical, for all $0^m 1^n$ -sequences \mathbf{b} (note that the even subalgebras of the Borels corresponding to various \mathbf{b} are fixed to be the same). On the other hand, we show in Section 5.3 that the canonical and dual canonical bases in $\mathbb{T}^{\mathbf{b}}$ for different \mathbf{b} are compatible and hence they are unique in a suitable sense; but the standard bases in $\mathbb{T}^{\mathbf{b}}$ do depend on \mathbf{b} . These results strongly support the existence of a possible \mathbb{Z} -grading lift in the sense of Soergel and others on the BGG category $\mathcal{O}_{\mathbf{b}}^{m|n}$ (cf. [BGS]). For the maximal parabolic category $\mathcal{F}^{m|n}$, the \mathbb{Z} -grading was established in [BS].

In another direction, a new approach to the full BGG category for the orthosymplectic Lie superalgebras is being developed by Huanchen Bao in his ongoing dissertation at University of Virginia. While fresh new ideas are needed in the new setting, the approach and the results established in the current paper have served as an inspiration and played a fundamental role.

1.7. Organization. The paper is organized as follows. It is divided into two parts. Part 1, which consists of Sections 2–5, deals with the combinatorics of Fock spaces, canonical bases, and BKL polynomials. Part 2, which consists of Sections 6–8, concerns the representation theory of $\mathfrak{gl}(m|n)$. The results of Part 1 are used in the final Section 8.

In Section 2, we review the quantum group, Hecke algebra of type A , and Jimbo duality. The Bruhat orderings on the Fock spaces $\mathbb{T}^{\mathbf{b}}$ are introduced.

In Section 3, we introduce the A - and B -completions of Fock spaces. Adapting from Lusztig [Lu2] and generalizing Brundan [Br1], we formulate a bar involution on the B -completions of Fock spaces, and establish the existence of canonical and dual canonical bases. The transition matrices between (dual) canonical bases and the standard monomial basis allows us to define the BKL polynomials.

In Section 4, via the notion of truncation maps, we compare the (dual) canonical bases on Fock spaces involving $\wedge^k \mathbb{V}$ or $\wedge^k \mathbb{W}$ for varying k . We formulate a combinatorial version of super duality, which is an isomorphism of Fock spaces preserving the (dual) canonical bases, when replacing $\wedge^{\infty} \mathbb{V}$ by $\wedge^{\infty} \mathbb{W}$. The presentation here generalizes and improves the special cases treated in [CWZ, CW1]. We then formulate a precise relationship between (dual) canonical bases of a Fock space and those of its various q -wedge subspaces, and thus a relationship between their BKL polynomials.

In Section 5, we develop a new approach to compare precisely the canonical as well as dual canonical bases in two Fock spaces associated to adjacent $0^m 1^n$ -sequences. For that purpose, we introduce two kinds of parabolic monomial bases denoted by N 's and U 's, the N 's being adapted to dual canonical basis while the U 's to the canonical basis. All these are built on computations on $\mathbb{V} \otimes \mathbb{W}$ and $\mathbb{W} \otimes \mathbb{V}$.

In Section 6, we show that the BGG category $\mathcal{O}_{\mathbf{b}}^{m|n}$ of $\mathfrak{gl}(m|n)$ -modules are independent of \mathbf{b} , and so denoted by $\mathcal{O}^{m|n}$. The \mathbf{b} -Verma modules and \mathbf{b} -tilting modules are introduced. We show that a \mathbf{b} -tilting module is always a \mathbf{b}' -tilting module for any other sequence \mathbf{b}' , and provide a precise identification when \mathbf{b} and \mathbf{b}' are adjacent. We also

introduce some auxiliary modules denoted by N 's and U 's in $\mathcal{O}^{m|n}$, which correspond to the parabolic monomial bases with same the letters of Section 5. The results of this section are valid for any basic Lie superalgebra.

In Section 7, we develop the super duality machinery in the generality we need (compare [CL, CLW]). We establish an equivalence of module categories for two infinite-rank Lie superalgebras \mathfrak{g} and $\check{\mathfrak{g}}$, which match the corresponding Kazhdan-Lusztig-Vogan polynomials in terms of Kostant \mathfrak{u} -homology groups. The category $\mathcal{O}^{m|n}$ (for varying finite m, n) and related parabolic categories are recovered via truncation functors.

In Section 8, we put together all the pieces from previous sections. We establish the compatibilities of the BKL conjectures for adjacent sequences (1.1), between full and parabolic BGG categories (1.2), as well as all remaining steps (1.3), (1.4) and (1.5). The BKL conjecture follows.

1.8. Notations. Let \mathcal{P} denote the set of partitions. For $\lambda \in \mathcal{P}$ we denote its conjugate and length by λ' and $\ell(\lambda)$, respectively. We let \mathbb{Z} , \mathbb{Z}_+ , \mathbb{Z}_- , and \mathbb{N} denote the sets of all, non-negative, non-positive, and positive integers, respectively. Denote by $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$. For $k \in \mathbb{N}$ we denote by $[k]$ the set $\{1, 2, \dots, k\}$. Similarly, we set $[\underline{k}] := \{\underline{1}, \underline{2}, \dots, \underline{k}\}$ and also write $[\infty]$ for $\{\underline{1}, \underline{2}, \underline{3}, \dots\}$. The symmetric group on a set of k elements is denoted by \mathfrak{S}_k .

Acknowledgments. The first author is partially supported by an NSC grant, and he thanks NCTS/TPE and the Department of Mathematics of University of Virginia for support. The second author is partially supported by an NSC grant, and he thanks NCTS/South for support. The third author is partially supported by an NSF grant DMS-1101268, and he thanks the Institute of Mathematics of Academia Sinica in Taiwan for providing excellent working environment and support.

Part 1. Combinatorics

2. FOCK SPACES AND BRUHAT ORDERINGS

In this section, we introduce the Fock space $\mathbb{T}^{\mathbf{b}}$, which is a tensor space of copies of the natural $U_q(\mathfrak{gl}_\infty)$ -module and its restricted dual, associated to a $0^m 1^n$ -sequence \mathbf{b} . We define the Bruhat ordering on $\mathbb{T}^{\mathbf{b}}$. Some q -wedge spaces are also introduced.

2.1. Quantum group. Let q be an indeterminate. The quantum group $U_q(\mathfrak{gl}_\infty)$ is defined to be the associative algebra over $\mathbb{Q}(q)$ generated by $E_a, F_a, K_a, K_a^{-1}, a \in \mathbb{Z}$,

subject to the following relations ($a, b \in \mathbb{Z}$):

$$\begin{aligned}
K_a K_a^{-1} &= K_a^{-1} K_a = 1, \\
K_a K_b &= K_b K_a, \\
K_a E_b K_a^{-1} &= q^{\delta_{a,b} - \delta_{a,b+1}} E_b, \\
K_a F_b K_a^{-1} &= q^{\delta_{a,b+1} - \delta_{a,b}} F_b, \\
E_a F_b - F_b E_a &= \delta_{a,b} \frac{K_{a,a+1} - K_{a+1,a}}{q - q^{-1}}, \\
E_a^2 E_b + E_b E_a^2 &= (q + q^{-1}) E_a E_b E_a, \quad \text{if } |a - b| = 1, \\
E_a E_b &= E_b E_a, \quad \text{if } |a - b| > 1, \\
F_a^2 F_b + F_b F_a^2 &= (q + q^{-1}) F_a F_b F_a, \quad \text{if } |a - b| = 1, \\
F_a F_b &= F_b F_a, \quad \text{if } |a - b| > 1.
\end{aligned}$$

Here $K_{a,a+1} := K_a K_{a+1}^{-1}$. For $r \geq 1$, we introduce the divided powers $E_a^{(r)} = E_a^r / [r]!$ and $F_a^{(r)} = F_a^r / [r]!$, where $[r] = (q^r - q^{-r}) / (q - q^{-1})$ and $[r]! = [1][2] \cdots [r]$.

Setting $\bar{q} = q^{-1}$ induces an automorphism on $\mathbb{Q}(q)$ denoted by $\bar{\cdot}$. Define the bar involution on $U_q(\mathfrak{gl}_\infty)$ to be the anti-linear automorphism $\bar{\cdot} : U_q(\mathfrak{gl}_\infty) \rightarrow U_q(\mathfrak{gl}_\infty)$ determined by $\bar{E}_a = E_a$, $\bar{F}_a = F_a$, and $\bar{K}_a = K_a^{-1}$. Here *anti-linear* means that $\overline{fu} = \bar{f}\bar{u}$, for $f \in \mathbb{Q}(q)$ and $u \in U_q(\mathfrak{gl}_\infty)$. The quantum group $U_q(\mathfrak{sl}_\infty)$ is the subalgebra generated by $\{E_a, F_a, K_{a,a+1} | a \in \mathbb{Z}\}$.

Let \mathbb{V} be the natural $U_q(\mathfrak{gl}_\infty)$ -module with basis $\{v_a\}_{a \in \mathbb{Z}}$ and $\mathbb{W} := \mathbb{V}^*$, the restricted dual module with basis $\{w_a\}_{a \in \mathbb{Z}}$ such that $\langle w_a, v_b \rangle = (-q)^{-a} \delta_{a,b}$. The actions of $U_q(\mathfrak{gl}_\infty)$ on \mathbb{V} and \mathbb{W} are given by the following formulas:

$$\begin{aligned}
K_a v_b &= q^{\delta_{a,b}} v_b, & E_a v_b &= \delta_{a+1,b} v_a, & F_a v_b &= \delta_{a,b} v_{a+1}, \\
K_a w_b &= q^{-\delta_{a,b}} w_b, & E_a w_b &= \delta_{a,b} w_{a+1}, & F_a w_b &= \delta_{a+1,b} w_a.
\end{aligned}$$

We shall use the co-multiplication Δ on $U_q(\mathfrak{gl}_\infty)$ defined by:

$$\begin{aligned}
\Delta(E_a) &= 1 \otimes E_a + E_a \otimes K_{a+1,a}, \\
\Delta(F_a) &= F_a \otimes 1 + K_{a,a+1} \otimes F_a, \\
\Delta(K_a) &= K_a \otimes K_a,
\end{aligned}$$

which restricts to a co-multiplication on $U_q(\mathfrak{sl}_\infty)$. Our Δ here is consistent with the one used by Kashiwara, but differs from [Lu2].

For $m, n \in \mathbb{Z}_+$, recall that $[m] = \{1, 2, \dots, m\}$, and denote the set of integer-valued functions on $[m+n]$ by \mathbb{Z}^{m+n} . We shall also identify f with the $(m+n)$ -tuple $(f(1), f(2), \dots, f(m+n))$ when convenient. Also, for a subset $I \subseteq [m+n]$, we shall denote the restriction of f to I by f_I . For example, if $I = \{i, i+1\}$, then the restriction of f to I will be denoted by $f_I = f_{i,i+1}$. This notation remains valid for functions defined on different domains as well.

2.2. Fock spaces. For $m, n \in \mathbb{Z}_+$, we let $\mathbf{b} = (b_1, b_2, \dots, b_{m+n})$ be a sequence of $m+n$ integers such that m of the b_i 's are equal to 0 and n of them are equal to 1. We call such a sequence a $0^m 1^n$ -sequence.

We associate to such a $0^m 1^n$ -sequence \mathbf{b} the following tensor space over $\mathbb{Q}(q)$, called the \mathbf{b} -Fock space or simply Fock space:

$$(2.1) \quad \mathbb{T}^{\mathbf{b}} := \mathbb{V}^{b_1} \otimes \mathbb{V}^{b_2} \otimes \cdots \otimes \mathbb{V}^{b_{m+n}},$$

where we denote

$$\mathbb{V}^{b_i} := \begin{cases} \mathbb{V}, & \text{if } b_i = 0, \\ \mathbb{W}, & \text{if } b_i = 1. \end{cases}$$

The tensors here and in similar settings later on are understood to be over the field $\mathbb{Q}(q)$. Note that the algebra $U_q(\mathfrak{gl}_\infty)$ acts on $\mathbb{T}^{\mathbf{b}}$ via the co-multiplication Δ .

Example 2.1. Let $m = 3$ and $n = 2$. Then $(0, 0, 1, 1, 0)$, $(0, 0, 0, 1, 1)$, and $(0, 1, 0, 1, 0)$ are $0^3 1^2$ -sequences. The associated $U_q(\mathfrak{gl}_\infty)$ -modules $\mathbb{T}^{\mathbf{b}}$ are respectively

$$\mathbb{V} \otimes \mathbb{V} \otimes \mathbb{W} \otimes \mathbb{W} \otimes \mathbb{V}, \quad \mathbb{V} \otimes \mathbb{V} \otimes \mathbb{V} \otimes \mathbb{W} \otimes \mathbb{W}, \quad \mathbb{V} \otimes \mathbb{W} \otimes \mathbb{V} \otimes \mathbb{W} \otimes \mathbb{V}.$$

For $f \in \mathbb{Z}^{m+n}$ we define

$$(2.2) \quad M_f^{\mathbf{b}} := \mathbf{v}_{f(1)}^{b_1} \otimes \mathbf{v}_{f(2)}^{b_2} \otimes \cdots \otimes \mathbf{v}_{f(m+n)}^{b_{m+n}},$$

where we use the notation $\mathbf{v}^{b_i} := \begin{cases} v, & \text{if } b_i = 0, \\ w, & \text{if } b_i = 1. \end{cases}$ We refer to $\{M_f^{\mathbf{b}} | f \in \mathbb{Z}^{m+n}\}$ as the *standard monomial basis* for $\mathbb{T}^{\mathbf{b}}$.

2.3. Bruhat ordering. Let \mathbf{P} be the free abelian group with orthonormal basis $\{\varepsilon_r | r \in \mathbb{Z}\}$ with respect to a bilinear form $(\cdot | \cdot)$. We define a partial order on \mathbf{P} by declaring $\nu \geq \mu$, for $\nu, \mu \in \mathbf{P}$, if $\nu - \mu$ is a non-negative integral linear combination of $\varepsilon_r - \varepsilon_{r+1}$, $r \in \mathbb{Z}$.

Fix a $0^m 1^n$ -sequence $\mathbf{b} = (b_1, \dots, b_{m+n})$. For $f \in \mathbb{Z}^{m+n}$ and $j \leq m+n$, we define

$$\text{wt}_{\mathbf{b}}^j(f) := \sum_{j \leq i} (-1)^{b_i} \varepsilon_{f(i)} \in \mathbf{P}, \quad \text{wt}_{\mathbf{b}}(f) := \text{wt}_{\mathbf{b}}^1(f) \in \mathbf{P}.$$

We define the *Bruhat ordering of type \mathbf{b}* on \mathbb{Z}^{m+n} , denoted by $\preceq_{\mathbf{b}}$, in terms of the partially ordered set (\mathbf{P}, \leq) as follows: $g \preceq_{\mathbf{b}} f$ if and only if $\text{wt}_{\mathbf{b}}(g) = \text{wt}_{\mathbf{b}}(f)$ and $\text{wt}_{\mathbf{b}}^j(g) \leq \text{wt}_{\mathbf{b}}^j(f)$, for all j . Note that when $n = 0$ this is simply the usual Bruhat ordering on the weight lattice \mathbb{Z}^m of $\mathfrak{gl}(m)$.

Following [Br1, §2-b] we introduce the following notation in our general setting:

$$(2.3) \quad \sharp_{\mathbf{b}}(f, a, j) := \sum_{j \leq i \leq [m+n], f(i) \leq a} (-1)^{b_i}, \quad \text{for } f \in \mathbb{Z}^{m+n}.$$

It is easy to see the following characterization of $\preceq_{\mathbf{b}}$ holds: for $g, f \in \mathbb{Z}^{m+n}$,

$$(2.4) \quad g \preceq_{\mathbf{b}} f \Leftrightarrow \sharp_{\mathbf{b}}(g, a, j) \leq \sharp_{\mathbf{b}}(f, a, j), \quad \forall a \in \mathbb{Z}, j \in [m+n], \text{ with equality for } j = 1.$$

For $i \in [m+n]$ we let $d_i \in \mathbb{Z}^{m+n}$ be determined by

$$(2.5) \quad d_i(j) = (-1)^{b_i} \delta_{ij}, \quad \text{for } j \in [m+n].$$

Let $f, g \in \mathbb{Z}^{m+n}$. We define

$$f \downarrow_{\mathbf{b}} g := \begin{cases} g = f \cdot (i, j), & \text{for } b_i = b_j = 0, i < j, f(i) > f(j), \\ g = f \cdot (i, j), & \text{for } b_i = b_j = 1, i < j, f(i) < f(j), \\ g = f - d_i + d_j, & \text{for } b_i \neq b_j, i < j, f(i) = f(j). \end{cases}$$

Here and further we denote the natural right action of \mathfrak{S}_{m+n} on \mathbb{Z}^{m+n} by $f \cdot \sigma := f \circ \sigma$, for $f \in \mathbb{Z}^{m+n}$ and $\sigma \in \mathfrak{S}_{m+n}$. The following lemma is clear.

Lemma 2.2. *Let $f, g \in \mathbb{Z}^{m+n}$. If there exists a sequence of elements $h_1, h_2, \dots, h_k \in \mathbb{Z}^{m+n}$ such that $h_i \downarrow_{\mathbf{b}} h_{i+1}$, for $1 \leq i \leq k-1$, with $h_1 = f$ and $h_k = g$, then $f \succeq_{\mathbf{b}} g$.*

Remark 2.3. In the case of standard $0^m 1^n$ -sequence $\mathbf{b}_{\text{st}} = (0, \dots, 0, 1, \dots, 1)$ as well as the opposite sequence $(1, \dots, 1, 0, \dots, 0)$, the converse of Lemma 2.2 holds, according to [Br1, Lemma 2.5]. However the converse is no longer true in general. Take the $0^3 1^2$ -sequence $\mathbf{b} = (0, 1, 0, 1, 0)$. Take $f = (4, 3, 5, 2, 1)$ and $g = (1, 2, 4, 3, 5)$. Then $f \succ_{\mathbf{b}} g$. But there exists no such sequence $\{h_i\}$ as in the above lemma moving f down to g .

The Bruhat ordering $\preceq_{\mathbf{b}}$ on \mathbb{Z}^{m+n} is defined to fit with the definition on the weight lattice of $\mathfrak{gl}(m|n)$ coming from central characters (see [CW2, Section 2.2]). But in this paper it would work if we have defined $\preceq_{\mathbf{b}}$ to be the transitive closure of $\downarrow_{\mathbf{b}}$.

The following lemma will be useful in the sequel.

Lemma 2.4. *The poset $(\mathbb{Z}^{m+n}, \preceq_{\mathbf{b}})$ satisfies the finite interval property. That is, given f, g with $g \preceq_{\mathbf{b}} f$, the set $\{h \in \mathbb{Z}^{m+n} | g \preceq_{\mathbf{b}} h \preceq_{\mathbf{b}} f\}$ is finite.*

Proof. We shall prove the precise and stronger statement that if $g \preceq_{\mathbf{b}} h \preceq_{\mathbf{b}} f$, then

$$(2.6) \quad |h(i)| \leq \max\{|f(j)|, |g(j)| \text{ with } j \in [m+n]\}, \quad \forall i \in [m+n].$$

We prove (2.6) by contradiction. Suppose that $|h(t)| > \max\{|f(j)|, |g(j)| \mid j \in [m+n]\}$, for some $t \in [m+n]$. Among the t 's with $|h(t)| = N$ maximal, we choose i as large as possible so that $h(i) = \pm N$. First suppose that $h(i) = N$. Then $\sharp_{\mathbf{b}}(f, N-1, i) = \sharp_{\mathbf{b}}(g, N-1, i)$. But clearly $\sharp_{\mathbf{b}}(h, N-1, i) \neq \sharp_{\mathbf{b}}(f, N-1, i)$. Now suppose that $h(i) = -N$. Then $0 = \sharp_{\mathbf{b}}(f, -N, i) = \sharp_{\mathbf{b}}(g, -N, i)$. But clearly $\sharp_{\mathbf{b}}(h, -N, i) = (-1)^{b_i}$. So in either case we cannot have $g \preceq_{\mathbf{b}} h \preceq_{\mathbf{b}} f$. \square

2.4. q -wedge spaces. For $k \in \mathbb{N}$, denote by \mathfrak{S}_k the symmetric group of permutations on $\{\underline{1}, \underline{2}, \dots, \underline{k}\}$. Let $\mathfrak{S}_{\infty} = \bigcup_k \mathfrak{S}_k$. Then \mathfrak{S}_k is generated by the simple transpositions $s_1 = (\underline{1}, \underline{2}), s_2 = (\underline{2}, \underline{3}), \dots, s_{k-1} = (\underline{k-1}, \underline{k})$.

The Iwahori-Hecke algebra associated to \mathfrak{S}_k (for $k \in \mathbb{N} \cup \{\infty\}$) is the associative $\mathbb{Q}(q)$ -algebra \mathcal{H}_k generated by H_i , $1 \leq i \leq k-1$, subject to the relations

$$\begin{aligned} (H_i - q^{-1})(H_i + q) &= 0, \\ H_i H_{i+1} H_i &= H_{i+1} H_i H_{i+1}, \\ H_i H_j &= H_j H_i, \quad \text{for } |i - j| > 1. \end{aligned}$$

Associated to $\sigma \in \mathfrak{S}_k$ with a reduced expression $\sigma = s_{i_1} \cdots s_{i_r}$, we define $H_{\sigma} := H_{i_1} \cdots H_{i_r}$. The bar involution on \mathcal{H}_k is the unique anti-linear automorphism defined

by $\overline{H_\sigma} = H_{\sigma^{-1}}^{-1}$, $\overline{q} = q^{-1}$, for all $\sigma \in \mathfrak{S}_k$. Set for $k \in \mathbb{N}$

$$(2.7) \quad H_0 := \sum_{\sigma \in \mathfrak{S}_k} (-q)^{\ell(\sigma) - \ell(w_0^{(k)})} H_\sigma,$$

where $w_0^{(k)}$ denotes the longest element in \mathfrak{S}_k . It is well known that (cf. [KL1], [So1, Proposition 2.9])

$$(2.8) \quad \overline{H_0} = H_0.$$

Now consider the tensor spaces $\mathbb{V}^{\otimes k}$ and $\mathbb{W}^{\otimes k}$, respectively. In either case we index the tensor factors by $[\underline{k}] := \{\underline{1}, \underline{2}, \dots, \underline{k}\}$. Now for an integer-valued function $f : [\underline{k}] \rightarrow \mathbb{Z}$, recall from (2.2) that $M_f = \mathbf{v}_{f(\underline{1})} \otimes \dots \otimes \mathbf{v}_{f(\underline{k})}$, where $\mathbf{v} = v$ for $\mathbb{V}^{\otimes k}$ and $\mathbf{v} = w$ for $\mathbb{W}^{\otimes k}$. The algebra \mathcal{H}_k acts on $\mathbb{V}^{\otimes k}$ and respectively on $\mathbb{W}^{\otimes k}$ on the right by

$$(2.9) \quad M_f H_i = \begin{cases} M_{f \cdot s_i}, & \text{if } f \prec_{\mathbf{b}} f \cdot s_i, \\ q^{-1} M_f, & \text{if } f = f \cdot s_i, \\ M_{f \cdot s_i} - (q - q^{-1}) M_f, & \text{if } f \succ_{\mathbf{b}} f \cdot s_i. \end{cases}$$

Here $\mathbf{b} = (0^k)$ for $\mathbb{V}^{\otimes k}$, and $\mathbf{b} = (1^k)$ for $\mathbb{W}^{\otimes k}$.

Lemma 2.5. [Jim] *The actions of $U_q(\mathfrak{gl}_\infty)$ and \mathcal{H}_k on the tensor space $\mathbb{V}^{\otimes k}$ (and respectively, on $\mathbb{W}^{\otimes k}$) commute with each other.*

Different commuting actions of $U_q(\mathfrak{gl}_\infty)$ and \mathcal{H}_k on $\mathbb{V}^{\otimes k}$ were used in [KMS] to construct the space $\wedge^k \mathbb{V}$ of finite q -wedges and then the space of infinite q -wedges by taking an appropriate limit $k \rightarrow \infty$. These spaces carry the action of $U_q(\mathfrak{gl}_\infty)$ (as a limiting case). The constructions in *loc. cit.* carry over using the above actions of $U_q(\mathfrak{gl}_\infty)$ and \mathcal{H}_k as we shall sketch below.

First consider the case when $k \in \mathbb{N}$. Following [KMS], we will regard $\wedge^k \mathbb{V}$ as the quotient of $\mathbb{V}^{\otimes k}$ by $\ker H_0$. Indeed $\ker H_0$ equals the sum of the kernels of the operators $H_i - q^{-1}$, $1 \leq i \leq k-1$, by [KMS, Proposition 1.2] (note that the Hecke algebra generator T_i used in *loc. cit.* corresponds to $-qH_i$.) We further denote by the q -wedges $v_{a_1} \wedge \dots \wedge v_{a_k}$ the image in $\wedge^k \mathbb{V}$ of $v_{a_1} \otimes \dots \otimes v_{a_k}$ under the canonical map. We have

$$(2.10) \quad \begin{aligned} \dots \wedge v_{a_i} \wedge v_{a_{i+1}} \wedge \dots &= -q^{-1}(\dots \wedge v_{a_{i+1}} \wedge v_{a_i} \wedge \dots), & \text{if } a_i < a_{i+1}; \\ \dots \wedge v_{a_i} \wedge v_{a_{i+1}} \wedge \dots &= 0, & \text{if } a_i = a_{i+1}. \end{aligned}$$

It follows that the elements $v_{a_1} \wedge \dots \wedge v_{a_k}$, where $a_1 > \dots > a_k$ form a basis for $\wedge^k \mathbb{V}$. By Lemma 2.5, $U_q(\mathfrak{gl}_\infty)$ acts naturally on $\wedge^k \mathbb{V}$.

Remark 2.6. For finite k , recalling that $\mathbb{V}^{\otimes k} = \ker H_0 \oplus \text{Im} H_0$, cf. [KMS, Proposition 1.1], we may regard equivalently $\wedge^k \mathbb{V}$ as the subspace $\text{Im} H_0$ of $\mathbb{V}^{\otimes k}$.

Now consider the limit $k \rightarrow \infty$. Let \mathbb{V}^∞ be the subspace of $\mathbb{V}^{\otimes \infty}$ spanned by vectors of the form

$$(2.11) \quad v_{p_1} \otimes v_{p_2} \otimes v_{p_3} \otimes \dots,$$

with $p_i = 1 - i$, for $i \gg 0$. Note that $U_q(\mathfrak{gl}_\infty)$ and the Hecke algebra act on this space. We define $\wedge^\infty \mathbb{V}$ to be the quotient of \mathbb{V}^∞ by the sum of the kernels of $H_i - q^{-1}$, for all

$i \geq 1$. The quantum group $U_q(\mathfrak{gl}_\infty)$ acts on $\wedge^\infty \mathbb{V}$ and this space has a basis given by the (formal) infinite q -wedges

$$v_{p_1} \wedge v_{p_2} \wedge v_{p_3} \wedge \cdots,$$

where $p_1 > p_2 > p_3 > \cdots$, and $p_i = 1 - i$ for $i \gg 0$. Here the infinite q -wedge is defined to be the image of the corresponding vector in (2.11) under the canonical quotient map. Alternatively, the space $\wedge^\infty \mathbb{V}$ has a basis indexed by partitions given by

$$|\lambda\rangle := v_{\lambda_1} \wedge v_{\lambda_2-1} \wedge v_{\lambda_3-2} \wedge \cdots,$$

where $\lambda = (\lambda_1, \lambda_2, \cdots)$ runs over the set \mathcal{P} of all partitions.

Let

$$(2.12) \quad \begin{aligned} \mathbb{Z}_+^k &= \{f : [\underline{k}] \rightarrow \mathbb{Z} \mid f(\underline{1}) > f(\underline{2}) > \cdots > f(\underline{k})\}, \text{ for } k \in \mathbb{N}, \\ \mathbb{Z}_+^\infty &= \{f : [\underline{\infty}] \rightarrow \mathbb{Z} \mid f(\underline{1}) > f(\underline{2}) > \cdots; f(\underline{t}) = 1 - t \text{ for } t \gg 0\}. \end{aligned}$$

For $f \in \mathbb{Z}_+^k$, we denote

$$\mathcal{V}_f = v_{f(\underline{1})} \wedge v_{f(\underline{2})} \wedge \cdots \wedge v_{f(\underline{k})}.$$

Then $\{\mathcal{V}_f \mid f \in \mathbb{Z}_+^k\}$ is a basis for $\wedge^k \mathbb{V}$, for $k \in \mathbb{N} \cup \{\infty\}$.

For $\underline{i} \in [\underline{k}]$, define $d_{\underline{i}} : [\underline{k}] \rightarrow \mathbb{Z}$ by letting $d_{\underline{i}}(j) = \delta_{ij}$, for $1 \leq j \leq k$, in the case of $\mathbb{V}^{\otimes k}$. Then $\wedge^k \mathbb{V}$ is naturally a $U_q(\mathfrak{gl}_\infty)$ -module, where the action of the Chevalley generators E_a, F_a, K_a , for $a \in \mathbb{Z}$, is given as follows:

$$(2.13) \quad \begin{aligned} E_a \mathcal{V}_f &= \begin{cases} \sum_{\underline{i}} \delta_{a+1, f(\underline{i})} \mathcal{V}_{f-d_{\underline{i}}}, & \text{if } f-d_{\underline{i}} \in \mathbb{Z}_+^k, \\ 0, & \text{otherwise.} \end{cases} \\ F_a \mathcal{V}_f &= \begin{cases} \sum_{\underline{i}} \delta_{a, f(\underline{i})} \mathcal{V}_{f+d_{\underline{i}}}, & \text{if } f+d_{\underline{i}} \in \mathbb{Z}_+^k, \\ 0, & \text{otherwise.} \end{cases} \\ K_a \mathcal{V}_f &= q^{\sum_{\underline{i}} \delta_{a, f(\underline{i})}} \mathcal{V}_f. \end{aligned}$$

Similar construction gives rise to $\wedge^k \mathbb{W}$, for $k \in \mathbb{N} \cup \{\infty\}$. The space $\wedge^\infty \mathbb{W}$ has a basis consisting of

$$w_{p_1} \wedge w_{p_2} \wedge w_{p_3} \wedge \cdots,$$

where $p_1 < p_2 < p_3 < \cdots$, and $p_i = i$, for $i \gg 0$. Alternatively, the space $\wedge^\infty \mathbb{W}$ has a basis indexed by partitions given by

$$|\lambda_*\rangle := w_{1-\lambda_1} \wedge w_{2-\lambda_2} \wedge w_{3-\lambda_3} \wedge \cdots,$$

where $\lambda = (\lambda_1, \lambda_2, \cdots)$ runs over the set of all partitions.

Let

$$(2.14) \quad \begin{aligned} \mathbb{Z}_-^k &= \{f : [\underline{k}] \rightarrow \mathbb{Z} \mid f(\underline{1}) < f(\underline{2}) < \cdots < f(\underline{k})\}, \text{ for } k \in \mathbb{N}, \\ \mathbb{Z}_-^\infty &= \{f : [\underline{\infty}] \rightarrow \mathbb{Z} \mid f(\underline{1}) < f(\underline{2}) < \cdots; f(\underline{t}) = t \text{ for } t \gg 0\}. \end{aligned}$$

For $f \in \mathbb{Z}_-^k$ we write

$$\mathcal{W}_f = w_{f(\underline{1})} \wedge w_{f(\underline{2})} \wedge \cdots \wedge w_{f(\underline{k})}.$$

Then $\{\mathcal{W}_f \mid f \in \mathbb{Z}_-^k\}$ is a basis for $\wedge^k \mathbb{W}$, for $k \in \mathbb{N} \cup \{\infty\}$.

For $\underline{i} \in [k]$, define $d_{\underline{i}} : [k] \rightarrow \mathbb{Z}$ by letting $d_{\underline{i}}(j) = -\delta_{ij}$, for $1 \leq j \leq k$, in the case of $\mathbb{W}^{\otimes k}$. Then $\wedge^k \mathbb{W}$ is naturally a $U_q(\mathfrak{gl}_{\infty})$ -module, where the action of the Chevalley generators E_a, F_a, K_a , for $a \in \mathbb{Z}$, on $\wedge^k \mathbb{W}$ is given as follows:

$$(2.15) \quad \begin{aligned} E_a \mathcal{W}_f &= \begin{cases} \sum_{\underline{i}} \delta_{a, f(\underline{i})} \mathcal{W}_{f-d_{\underline{i}}}, & \text{if } f - d_{\underline{i}} \in \mathbb{Z}_{-}^k, \\ 0, & \text{otherwise.} \end{cases} \\ F_a \mathcal{W}_f &= \begin{cases} \sum_{\underline{i}} \delta_{a+1, f(\underline{i})} \mathcal{W}_{f+d_{\underline{i}}}, & \text{if } f + d_{\underline{i}} \in \mathbb{Z}_{-}^k, \\ 0, & \text{otherwise.} \end{cases} \\ K_a \mathcal{W}_f &= q^{-\sum_{\underline{i}} \delta_{a, f(\underline{i})}} \mathcal{W}_f. \end{aligned}$$

We define a $\mathbb{Q}(q)$ -linear isomorphism $\mathfrak{h} : \wedge^{\infty} \mathbb{V} \rightarrow \wedge^{\infty} \mathbb{W}$ by

$$\mathfrak{h}(|\lambda\rangle) := |\lambda'_*\rangle, \quad \text{for } \lambda \in \mathcal{P}.$$

Proposition 2.7. [CWZ, Theorem 6.3] *The map $\mathfrak{h} : \wedge^{\infty} \mathbb{V} \rightarrow \wedge^{\infty} \mathbb{W}$ is an isomorphism of $U_q(\mathfrak{sl}_{\infty})$ -modules (both are isomorphic to the basic module of $U_q(\mathfrak{sl}_{\infty})$).*

3. CANONICAL BASES AND BRUNDAN-KAZHDAN-LUSZTIG POLYNOMIALS

In this section, we introduce the A - and B -completions of $\mathbb{T}^{\mathbf{b}}$. Then we define bar-involution, canonical and dual canonical bases in the B -completion of $\mathbb{T}^{\mathbf{b}}$. The Brundan-Kazhdan-Lusztig polynomials are also introduced.

3.1. Quasi- \mathcal{R} -matrix. Let M be a $U_q(\mathfrak{gl}_{\infty})$ -module equipped with a $\mathbb{Q}(q)$ -anti-linear bar involution $\bar{\cdot} : M \rightarrow M$, such that $\overline{um} = \bar{u}\bar{m}$, for all $u \in U_q(\mathfrak{gl}_{\infty})$ and $m \in M$. Suppose furthermore that M has a basis B consisting of bar-invariant weight vectors. We shall refer to (M, B) or simply M as a *weakly based module*. We note that Lusztig introduced the notion of a based module in [Lu2, 27.1.2], which is a weakly based module satisfying additional conditions.

Example 3.1. The $U_q(\mathfrak{gl}_{\infty})$ -modules \mathbb{V} and \mathbb{W} have bar involutions defined by $\overline{v_a} = v_a$ and $\overline{w_a} = w_a$, respectively, that are compatible with the actions of the quantum group. Thus, $(\mathbb{V}, \mathbb{B}^0)$ and $(\mathbb{W}, \mathbb{B}^1)$ are weakly based modules, where we denote $\mathbb{B}^0 = \{v_a | a \in \mathbb{Z}\}$ and $\mathbb{B}^1 = \{w_a | a \in \mathbb{Z}\}$. It follows from (2.13) and (2.15) that the $U_q(\mathfrak{gl}_{\infty})$ -modules $\wedge^k \mathbb{V}$ and $\wedge^k \mathbb{W}$ are also weakly based modules with basis given by $\{\mathcal{V}_f | f \in \mathbb{Z}_{+}^k\}$, and $\{\mathcal{W}_f | f \in \mathbb{Z}_{-}^k\}$, respectively, for $k \in \mathbb{N}$. The same is true for $k = \infty$ so that $(\wedge^{\infty} \mathbb{V}, \{|\lambda\rangle | \lambda \in \mathcal{P}\})$ and $(\wedge^{\infty} \mathbb{W}, \{|\lambda_*\rangle | \lambda \in \mathcal{P}\})$ are also weakly based modules. (Actually these are all examples of based modules in the sense of Lusztig, but we will not need this fact.)

In what follows we shall apply results from [Lu2] and [Jan]. To translate their results to our setting, we need to replace q^{-1} therein by q , and interchange E_a with F_a , for all $a \in \mathbb{Z}$, in order to match our co-multiplication with theirs. From Lusztig's theory of based modules [Lu2, Chapter 27], using the quasi- \mathcal{R} -matrix Θ , one can construct from k weakly based modules (M_i, B_i) two distinguished bases of the $U_q(\mathfrak{gl}_{\infty})$ -module $M_1 \otimes M_2 \otimes \cdots \otimes M_k$, called canonical and dual canonical basis, respectively. We shall review and extend these constructions below, as strictly speaking Lusztig's construction was carried out for finite-rank quantum groups.

In order to construct a bar involution on the tensor product of two weakly based modules, we will first define the quasi- \mathcal{R} -matrix Θ , which in turn is based on the existence of a PBW-type basis.

Denote by Φ^+ the standard positive root system of $U_q(\mathfrak{gl}_\infty)$, and set

$$P^+ = \sum_{\alpha \in \Phi^+} \mathbb{Z}_+ \alpha.$$

For $k \in \mathbb{N}$, let $U_q(\mathfrak{gl}_{|k|})$ be the subalgebra of $U_q(\mathfrak{gl}_\infty)$ generated by $\{E_a, F_a, K_a^{\pm 1}, K_{a+1}^{\pm 1}\}$ for $-k \leq a \leq k-1$. Then we have $U_q(\mathfrak{gl}_{|k|}) \subseteq U_q(\mathfrak{gl}_{|k+1|})$ and $\bigcup_k U_q(\mathfrak{gl}_{|k|}) = U_q(\mathfrak{gl}_\infty)$. Furthermore, $\mathcal{U}_{|k|}^\pm \subseteq \mathcal{U}_{|k+1|}^\pm$ and $\bigcup_k \mathcal{U}_{|k|}^\pm = \mathcal{U}^\pm$, where $\mathcal{U}_{|k|}^\pm$ and \mathcal{U}^\pm denote the positive and negative parts of $U_q(\mathfrak{gl}_{|k|})$ and $U_q(\mathfrak{gl}_\infty)$, respectively.

For $k \in \mathbb{N}$, let $\mathfrak{S}_{|k|}$ denote the symmetric group on the set $\{-k, -k+1, \dots, 0, 1, \dots, k\}$, and let $w_0^{|k|}$ denote the longest element in $\mathfrak{S}_{|k|}$. Then there exists a reduced expression $w' \in \mathfrak{S}_{|k+1|}$ such that

$$w_0^{|k+1|} = w_0^{|k|} w', \quad \text{where } \ell(w_0^{|k+1|}) = \ell(w_0^{|k|}) + \ell(w').$$

Hence there exists an infinite sequence of simple roots $\alpha_1, \alpha_2, \alpha_3, \dots$ such that for each k we have a reduced expression for $w_0^{|k|}$ as

$$(3.1) \quad w_0^{|k|} = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_N}, \quad \text{where } N = k(2k+1).$$

Associated to a simple root α one can define an automorphism $T_\alpha : U_q(\mathfrak{gl}_\infty) \rightarrow U_q(\mathfrak{gl}_\infty)$ [Jan, 8.14]. For a sequence of non-negative integers (a_i) indexed by $[N]$, we define the element

$$(3.2) \quad T_{\alpha_1} T_{\alpha_2} \cdots T_{\alpha_{N-1}} (E_{\alpha_N}^{a_N}) \cdots T_{\alpha_1} T_{\alpha_2} (E_{\alpha_3}^{a_3}) \cdot T_{\alpha_1} (E_{\alpha_2}^{a_2}) \cdot E_{\alpha_1}^{a_1} \in U_q(\mathfrak{gl}_{|k|}).$$

Then, the set of all such elements form a basis for the positive part $\mathcal{U}_{|k|}^+$ [Jan, Theorem 8.24]. Taking the limit $k \rightarrow \infty$ we obtain a basis for \mathcal{U}^+ consisting of elements (3.2) with N arbitrarily large. Replacing the E_{α_i} 's in (3.2) by the corresponding F_{α_i} 's, we obtain a basis for \mathcal{U}^- . For $\mu \in P^+$, denote by \mathcal{U}_μ^+ the corresponding μ -weight space of \mathcal{U}^+ , and by $\mathcal{U}_{-\mu}^-$ the corresponding $(-\mu)$ -weight space of \mathcal{U}^- .

The quasi- \mathcal{R} -matrix Θ is an element in some suitable completion of $\mathcal{U}^+ \otimes \mathcal{U}^-$. For later use let us write down an explicit formula for Θ by mimicking the construction in [Jan, 8.30(2)]. Associated to the positive root $s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_{t-1}}(\alpha_t)$ for $t \in \mathbb{N}$, we define

$$(3.3) \quad \Theta_{[t]} := \sum_{r \geq 0} q^{r(r-1)/2} \frac{(q - q^{-1})^r}{[r]!} T_{\alpha_1} \cdots T_{\alpha_{t-1}} (E_{\alpha_t}^r) \otimes T_{\alpha_1} \cdots T_{\alpha_{t-1}} (F_{\alpha_t}^r).$$

Now let $\mu \in P^+$. We choose k large enough so that μ is a weight of $\mathfrak{gl}_{|k|}$. We set $\Theta_\mu \otimes \Theta_{-\mu}$ to be the $\mathcal{U}_\mu^+ \otimes \mathcal{U}_{-\mu}^-$ -component of the product $\Theta_{[N]} \cdots \Theta_{[2]} \Theta_{[1]}$. This definition is independent of sufficiently large k and N , and we define the quasi- \mathcal{R} -matrix Θ for $U_q(\mathfrak{gl}_\infty)$ as

$$(3.4) \quad \Theta = \sum_{\mu \in P^+} \Theta_\mu \otimes \Theta_{-\mu}.$$

Formally, we have just made sense of the infinite product $\Theta = \cdots \Theta_{[3]} \Theta_{[2]} \Theta_{[1]}$. Similarly, the quasi- \mathcal{R} -matrix $\Theta^{(k)}$ for $U_q(\mathfrak{gl}_{|k|})$ is defined as [Jan, Chapter 8]

$$(3.5) \quad \Theta^{(k)} = \Theta_{[N]} \cdots \Theta_{[3]} \Theta_{[2]} \Theta_{[1]}, \quad \text{where } N = k(2k+1).$$

3.2. Completions. Let \mathbf{b} be a fixed $0^m 1^n$ -sequence. Let $k \in \mathbb{N}$ and let $\mathbb{T}_{\leq |k|}^{\mathbf{b}}$ be the (truncated) $\mathbb{Q}(q)$ -subspace of $\mathbb{T}^{\mathbf{b}}$, spanned by the elements $M_f^{\mathbf{b}}$ defined in (2.2), with $-k \leq f(i) \leq k$, for all $i \in [m+n]$. Let

$$(3.6) \quad \pi_k : \mathbb{T}^{\mathbf{b}} \longrightarrow \mathbb{T}_{\leq |k|}^{\mathbf{b}}$$

be the natural projection map with respect to the basis $\{M_f^{\mathbf{b}}\}$ for $\mathbb{T}^{\mathbf{b}}$. The kernels of the π_k 's define a linear topology on the vector space $\mathbb{T}^{\mathbf{b}}$. We then let $\tilde{\mathbb{T}}^{\mathbf{b}}$ be the completion of $\mathbb{T}^{\mathbf{b}}$ with respect to this topology. Formally, every element in $\tilde{\mathbb{T}}^{\mathbf{b}}$ is a possibly infinite linear combination of M_f , for $f \in \mathbb{Z}^{m+n}$. We let $\hat{\mathbb{T}}^{\mathbf{b}}$ denote the subspace of $\tilde{\mathbb{T}}^{\mathbf{b}}$ spanned by elements of the form

$$(3.7) \quad M_f + \sum_{g \prec_{\mathbf{b}} f} r_g M_g, \quad \text{for } r_g \in \mathbb{Q}(q).$$

Definition 3.2. The $\mathbb{Q}(q)$ -vector spaces $\tilde{\mathbb{T}}^{\mathbf{b}}$ and $\hat{\mathbb{T}}^{\mathbf{b}}$ are called the *A-completion* and *B-completion* of $\mathbb{T}^{\mathbf{b}}$, respectively.

Remark 3.3. A similar completion was introduced by Brundan [Br1, §2-d] for the standard $0^m 1^n$ -sequence $\mathbf{b}_{\text{st}} = (\underbrace{0, \dots, 0}_m, \underbrace{1, \dots, 1}_n)$.

3.3. Bar involution. For two finite-dimensional weakly based modules (M, B) and (N, C) of a finite-rank quantum group, Lusztig [Lu2, 27.3.1] defined a bar map ψ on the tensor space $M \otimes N$ via the quasi- \mathcal{R} -matrix Θ by

$$(3.8) \quad \psi(m \otimes n) := \Theta(\overline{m} \otimes \overline{n}), \quad \forall m \in M, n \in N.$$

Then, the bar map ψ is an involution by [Lu2, Corollary 4.1.3] and furthermore is compatible with the action on $M \otimes N$ induced by the co-multiplication Δ by [Lu2, Lemma 24.1.2]. We shall adapt this construction to $U_q(\mathfrak{gl}_{\infty})$ -modules below. However, because our modules are not finite dimensional, we shall need to deal with completion issues.

Consider the weakly based modules $(\mathbb{V}, \mathbb{B}^0)$ and $(\mathbb{W}, \mathbb{B}^1)$. Let $\mathbf{b} = (b_1, b_2, \dots, b_{m+n})$ be a fixed $0^m 1^n$ -sequence. Let $\mathbb{T}^{\mathbf{b}}$ be as in (2.1) and recall its *A-completion* $\tilde{\mathbb{T}}^{\mathbf{b}}$ from Definition 3.2. We first construct a $\mathbb{Q}(q)$ -anti-linear bar map $\bar{\cdot} : \mathbb{T}^{\mathbf{b}} \rightarrow \tilde{\mathbb{T}}^{\mathbf{b}}$. To be definite for now, we regard $\mathbb{T}^{\mathbf{b}}$ in (2.1) as taking tensor product successively from left to right. By (3.8) we can use the quasi- \mathcal{R} -matrix $\Theta^{(k)}$ to construct an involution $\psi^{(k)}$ on $\mathbb{T}_{\leq |k|}^{\mathbf{b}}$, which is a tensor product of $U_q(\mathfrak{gl}_{|k|})$ -modules, for $k \in \mathbb{N}$. Recall the projection map $\pi_k : \mathbb{T}^{\mathbf{b}} \rightarrow \mathbb{T}_{\leq |k|}^{\mathbf{b}}$.

Lemma 3.4. *Let $f \in \mathbb{Z}^{m+n}$ and $k \in \mathbb{N}$ be such that $|f(i)| \leq k$, for all $i \in [m+n]$. Then we have $M_f \in \mathbb{T}_{\leq |k|}^{\mathbf{b}}$, and*

$$(3.9) \quad \psi^{(k)}(M_f) = \pi_k \left(\psi^{(\ell)}(M_f) \right), \quad \text{for } \ell \geq k.$$

Proof. It is clear that $M_f \in \mathbb{T}_{\leq |k|}^{\mathbf{b}}$. We prove (3.9) by induction on $m+n$. The case for $m+n \leq 2$ is easily checked directly. For $\ell \geq k$ we compute that

$$\begin{aligned} & \pi_k(\psi^{(\ell)} M_f) \\ &= \pi_k \left(\Theta^{(\ell)} \left(\psi^{(\ell)}(\mathbf{v}_{f(1)}^{b_1} \otimes \mathbf{v}_{f(2)}^{b_2} \otimes \cdots \otimes \mathbf{v}_{f(m+n-1)}^{b_{m+n-1}}) \otimes \mathbf{v}_{f(m+n)}^{b_{m+n}} \right) \right) \\ &= \pi_k \left(\sum_{\mu} \Theta_{\mu}^{(\ell)} \left(\psi^{(\ell)}(\mathbf{v}_{f(1)}^{b_1} \otimes \mathbf{v}_{f(2)}^{b_2} \otimes \cdots \otimes \mathbf{v}_{f(m+n-1)}^{b_{m+n-1}}) \right) \otimes \Theta_{-\mu}^{(\ell)} \left(\mathbf{v}_{f(m+n)}^{b_{m+n}} \right) \right) \\ &= \pi_k \left(\sum_{\mu} \Theta_{\mu}^{(k)} \left(\psi^{(\ell)}(\mathbf{v}_{f(1)}^{b_1} \otimes \mathbf{v}_{f(2)}^{b_2} \otimes \cdots \otimes \mathbf{v}_{f(m+n-1)}^{b_{m+n-1}}) \right) \otimes \Theta_{-\mu}^{(k)} \left(\mathbf{v}_{f(m+n)}^{b_{m+n}} \right) \right) \\ &= \pi_k \left(\sum_{\mu} \Theta_{\mu}^{(k)} \pi_k \left(\psi^{(\ell)}(\mathbf{v}_{f(1)}^{b_1} \otimes \mathbf{v}_{f(2)}^{b_2} \otimes \cdots \otimes \mathbf{v}_{f(m+n-1)}^{b_{m+n-1}}) \right) \otimes \Theta_{-\mu}^{(k)} \left(\mathbf{v}_{f(m+n)}^{b_{m+n}} \right) \right) \\ &\stackrel{(*)}{=} \pi_k \left(\sum_{\mu} \Theta_{\mu}^{(k)} \left(\psi^{(k)}(\mathbf{v}_{f(1)}^{b_1} \otimes \mathbf{v}_{f(2)}^{b_2} \otimes \cdots \otimes \mathbf{v}_{f(m+n-1)}^{b_{m+n-1}}) \right) \otimes \Theta_{-\mu}^{(k)} \left(\mathbf{v}_{f(m+n)}^{b_{m+n}} \right) \right) \\ &= \pi_k \left(\psi^{(k)} \left((\mathbf{v}_{f(1)}^{b_1} \otimes \mathbf{v}_{f(2)}^{b_2} \otimes \cdots \otimes \mathbf{v}_{f(m+n-1)}^{b_{m+n-1}}) \otimes \mathbf{v}_{f(m+n)}^{b_{m+n}} \right) \right) \\ &= \psi^{(k)}(M_f). \end{aligned}$$

The third identity above uses the fact that $\pi_k(y \otimes \Theta^{(\ell)}(\mathbf{v}_t^{b_{m+n}})) = \pi_k(y \otimes \Theta^{(k)}(\mathbf{v}_t^{b_{m+n}}))$, for any $y \in \mathbb{T}^{\mathbf{b}'}$ with $\mathbf{b}' = (b_1, \dots, b_{m+n-1})$ and $|t| \leq k$, and the induction hypothesis is used in the identity (*). \square

It follows that the element $\lim_{\ell \rightarrow \infty} \psi^{(\ell)}(M_f)$, for any $f \in \mathbb{Z}^{m+n}$, is a well-defined element in $\tilde{\mathbb{T}}^{\mathbf{b}}$. We define

$$(3.10) \quad \psi(M_f) := \lim_{\ell \rightarrow \infty} \psi^{(\ell)}(M_f).$$

It follows immediately from (3.9) that $\psi(M_f) \in \tilde{\mathbb{T}}^{\mathbf{b}}$ and $\psi^{(k)}(M_f) = \pi_k(\psi(M_f))$.

A different tensor order on $\mathbb{T}^{\mathbf{b}}$ and hence on $\mathbb{T}_{\leq |k|}^{\mathbf{b}}$ would give a different inductive way to define a map ${}'\psi^{(k)}$, similar to $\psi^{(k)}$, for $k \in \mathbb{N}$. For $f \in \mathbb{Z}^{m+n}$, choose ℓ so that $\ell \geq \max_i \{|f(i)|\}$. By [Lu2, 4.2.4] we have $\psi^{(\ell)}(M_f) = {}'\psi^{(\ell)}(M_f)$, and hence $\pi_k(\psi^{(\ell)}(M_f)) = \pi_k({}'\psi^{(\ell)}(M_f))$ whenever $\ell \geq k$. Thus, $\lim_{\ell \rightarrow \infty} {}'\psi^{(\ell)} M_f$ is a well-defined element in $\tilde{\mathbb{T}}^{\mathbf{b}}$, and it coincides with $\lim_{\ell \rightarrow \infty} \psi^{(\ell)} M_f$. Hence we have proved the following.

Proposition 3.5. *The bar map $\bar{\cdot}: \mathbb{T}^{\mathbf{b}} \rightarrow \tilde{\mathbb{T}}^{\mathbf{b}}$, given by $\overline{M}_f = \psi(M_f)$ (see (3.10)), is well-defined, namely it is independent of the tensor order on $\mathbb{T}^{\mathbf{b}}$.*

For $m = 2$, $n = 2$ and $\mathbf{b} = (0, 1, 0, 1)$ we may regard $\mathbb{T}^{\mathbf{b}} = ((\mathbb{V} \otimes \mathbb{W}) \otimes \mathbb{V}) \otimes \mathbb{W}$, and apply the quasi- \mathcal{R} -matrix Θ repeatedly from left to right and get a bar map on $\mathbb{T}^{\mathbf{b}}$ as above. We can also regard $\mathbb{T}^{\mathbf{b}}$ as $(\mathbb{V} \otimes (\mathbb{W} \otimes \mathbb{V})) \otimes \mathbb{W}$, and use this order to get a bar map. Proposition 3.5 says that the two bar maps coincide. Recall the B -completion $\widehat{\mathbb{T}}^{\mathbf{b}}$ from Definition 3.2.

Proposition 3.6. *Let $f \in \mathbb{Z}^{m+n}$ and $M_f \in \mathbb{T}^{\mathbf{b}}$. We have*

$$\overline{M}_f = M_f + \sum_{g \prec_{\mathbf{b}} f} r_{gf}(q) M_g,$$

where $r_{gf}(q) \in \mathbb{Z}[q, q^{-1}]$ and the sum is possibly infinite. Hence, we have $\bar{\cdot}: \mathbb{T}^{\mathbf{b}} \rightarrow \widehat{\mathbb{T}}^{\mathbf{b}}$.

Proof. By (3.9) we have $\overline{M}_f \in \widehat{\mathbb{T}}^{\mathbf{b}}$. Making use of the explicit form (3.4) of the quasi- \mathcal{R} -matrix Θ , we first observe that the proposition holds in the cases when $m + n \leq 2$. We now proceed by induction on $m + n$.

Let $f \in \mathbb{Z}^{m+n}$ and set $f' = f_{[m+n-1]}$. Furthermore, for $\mathbf{b} = (b_1, b_2, \dots, b_{m+n})$, we set $\mathbf{b}' = (b_1, b_2, \dots, b_{m+n-1})$. We have

$$M_f = \mathbf{v}_{f(1)}^{b_1} \otimes \mathbf{v}_{f(2)}^{b_2} \otimes \cdots \otimes \mathbf{v}_{f(m+n)}^{b_{m+n}}.$$

By inductive assumption we compute that

$$\begin{aligned} \overline{M}_f &= \Theta \left(\overline{\mathbf{v}_{f(1)}^{b_1} \otimes \mathbf{v}_{f(2)}^{b_2} \otimes \cdots \otimes \mathbf{v}_{f(m+n-1)}^{b_{m+n-1}} \otimes \mathbf{v}_{f(m+n)}^{b_{m+n}}} \right) \\ &= \sum_{\mu} \Theta_{\mu} \left(\overline{\mathbf{v}_{f(1)}^{b_1} \otimes \mathbf{v}_{f(2)}^{b_2} \otimes \cdots \otimes \mathbf{v}_{f(m+n-1)}^{b_{m+n-1}}} \right) \otimes \Theta_{-\mu} \left(\mathbf{v}_{f(m+n)}^{b_{m+n}} \right) \\ &= \sum_{\mu} \Theta_{\mu} \left(M_{f'} + \sum_{g' \prec_{\mathbf{b}'} f'} s_{g'f'}(q) M_{g'} \right) \otimes \Theta_{-\mu} \left(\mathbf{v}_{f(m+n)}^{b_{m+n}} \right), \end{aligned}$$

where $s_{g'f'}(q) \in \mathbb{Z}[q, q^{-1}]$. Now recall that $\Theta_{\mu} \otimes \Theta_{-\mu}$ is a $\mathbb{Q}(q)$ -linear combination of products of the form $T_{\alpha_1} \cdots T_{\alpha_{k-1}}(E_{\alpha_k}^r) \otimes T_{\alpha_1} \cdots T_{\alpha_{k-1}}(F_{\alpha_k}^r)$. From the explicit formulas for these expressions in [Jan, 8.14(7)] and the cases with $m + n = 2$, it follows that such an element, when applied to an element of the form $M_h \otimes \mathbf{v}_b^{b_{m+n}}$, for $h \in \mathbb{Z}^{m+n-1}$ and $b \in \mathbb{Z}$, gives a $\mathbb{Q}(q)$ -linear combination of elements of the form $M_t \otimes \mathbf{v}_c^{b_{m+n}}$, for $t \in \mathbb{Z}^{m+n-1}$ and $c \in \mathbb{Z}$; moreover we have a sequence of weights (h_i, c_i) , $i = 1, \dots, k$, such that

$$(h, b) = (h_1, c_1) \downarrow_{\mathbf{b}} (h_2, c_2) \downarrow_{\mathbf{b}} \cdots \downarrow_{\mathbf{b}} (h_k, c_k) = (t, c).$$

By Lemma 2.2, we have $(t, c) \preceq_{\mathbf{b}} (h, b)$, and hence $\overline{M}_f = M_f + \sum_{g \prec_{\mathbf{b}} f} r_{gf}(q) M_g$, for $r_{gf}(q) \in \mathbb{Q}(q)$.

It remains to show that $r_{gf}(q) \in \mathbb{Z}[q, q^{-1}]$. For this we first observe that the $\mathbb{Z}[q, q^{-1}]$ -span of the standard monomial basis elements in $\mathbb{T}^{\mathbf{b}}$ is invariant under the action of $K_a^{\pm 1}$, and the divided powers $E_a^{(j)}$ and $F_a^{(j)}$, for $a \in \mathbb{Z}$ and $j \in \mathbb{N}$. From this observation, [Jan, 8.14(7)], and formula (3.3) for $\Theta_{[t]}$, it follows that $r_{gf}(q) \in \mathbb{Z}[q, q^{-1}]$. \square

As we have already noted, due to the infinite-dimensionality of \mathbb{V} and \mathbb{W} , the bar involution does not preserve the space $\mathbb{T}^{\mathbf{b}}$. However, we have the following.

Lemma 3.7. *The bar map $\bar{\cdot}: \mathbb{T}^{\mathbf{b}} \rightarrow \widehat{\mathbb{T}}^{\mathbf{b}}$ extends to $\bar{\cdot}: \widehat{\mathbb{T}}^{\mathbf{b}} \rightarrow \widehat{\mathbb{T}}^{\mathbf{b}}$. Furthermore, the bar map on $\widehat{\mathbb{T}}^{\mathbf{b}}$ is an involution.*

Proof. To show that the bar map extends to $\widehat{\mathbb{T}}^{\mathbf{b}}$ we need to show that if $y = M_f + \sum_{g \prec_{\mathbf{b}} f} r_g(q) M_g \in \widehat{\mathbb{T}}^{\mathbf{b}}$, $r_g(q) \in \mathbb{Q}(q)$, then $\bar{y} \in \widehat{\mathbb{T}}^{\mathbf{b}}$. By Proposition 3.6 and the definition of $\widehat{\mathbb{T}}^{\mathbf{b}}$, it remains to show that $\bar{y} \in \widetilde{\mathbb{T}}^{\mathbf{b}}$. To see this, we note that if the coefficient of M_h in \bar{y} is nonzero, then there exists $g \preceq_{\mathbf{b}} f$ such that $r_{hg}(q) \neq 0$. Thus we have $h \preceq_{\mathbf{b}} g \preceq_{\mathbf{b}} f$. However, by Lemma 2.4 there are only finitely many such g 's. Thus, only finitely many g 's can contribute to the coefficient of M_h in \bar{y} , and hence $\bar{y} \in \widetilde{\mathbb{T}}^{\mathbf{b}}$.

To show that $\bar{\cdot}$ is an involution, we need to show that for fixed $f, g \in \mathbb{Z}^{m+n}$ with $g \preceq_{\mathbf{b}} f$ we have

$$(3.11) \quad \sum_{g \preceq_{\mathbf{b}} h \preceq_{\mathbf{b}} f} r_{gh}(q) \overline{r_{hf}(q)} = \delta_{fg}.$$

By Lemma 2.4, there are only finitely many such h 's with $g \preceq_{\mathbf{b}} h \preceq_{\mathbf{b}} f$. This together with Lemma 3.4 implies that (3.11) is equivalent to the same identity on the finite-dimensional space $\mathbb{T}_{\leq |k|}^{\mathbf{b}}$, for $k \gg 0$. But in this case [Lu2, Corollary 4.1.3] is applicable. So we conclude that (3.11) holds and so the bar map is an involution. \square

3.4. Canonical basis. For $r(q) \in \mathbb{Q}(q)$ recall that $\overline{r(q)} = r(q^{-1})$. A version of the following lemma goes back to [KL1]. We note that [Lu2, Lemma 24.2.1] is stated in a slightly different form, and also that, although (vi)–(viii) are not listed there, the same proof therein can be used to establish them.

Lemma 3.8. [Lu2, Lemma 24.2.1] *Let (I, \preceq) be a partially ordered set satisfying the finite interval property. Assume that for every $i \preceq j$ we are given elements $r_{ij} \in \mathbb{Z}[q, q^{-1}]$ such that*

- (i) $r_{ii} = 1$, for all $i \in I$,
- (ii) $\sum_{h, i \preceq h \preceq j} r_{ih} \overline{r_{hj}} = \delta_{ij}$.

Then there exists a unique family of elements $t_{ij} \in \mathbb{Z}[q]$ for all $i \preceq j$ such that

- (iii) $t_{ii} = 1$, for all $i \in I$,
- (iv) $t_{ij} \in q\mathbb{Z}[q]$, for all $i \prec j$,
- (v) $t_{ij} = \sum_{h, i \preceq h \preceq j} r_{ih} \overline{t_{hj}}$, for all $i \preceq j$.

Furthermore, there exists a unique family of elements $\ell_{ij} \in \mathbb{Z}[q^{-1}]$ for all $i \preceq j$ such that

- (vi) $\ell_{ii} = 1$, for all $i \in I$,
- (vii) $\ell_{ij} \in q^{-1}\mathbb{Z}[q^{-1}]$, for all $i \prec j$,
- (viii) $\ell_{ij} = \sum_{h, i \preceq h \preceq j} r_{ih} \overline{\ell_{hj}}$, for all $i \preceq j$.

We shall now apply Lemma 3.8 to the partially ordered set $(\mathbb{Z}^{m+n}, \preceq_{\mathbf{b}})$. Note first that the finite interval condition in Lemma 3.8 is satisfied due to Lemma 2.4. Recall from Proposition 3.6 that $\overline{M}_f = M_f + \sum_{g \prec_{\mathbf{b}} f} r_{gf}(q) M_g$. So Property (i) is clear, and (ii) follows readily by applying the anti-linear bar-involution $\bar{\cdot}$ in Lemma 3.7 to the above identity. Hence we have established the following.

Proposition 3.9. *The $\mathbb{Q}(q)$ -vector space $\widehat{\mathbb{T}}^{\mathbf{b}}$ has unique bar-invariant topological bases*

$$\{T_f^{\mathbf{b}} | f \in \mathbb{Z}^{m+n}\} \text{ and } \{L_f^{\mathbf{b}} | f \in \mathbb{Z}^{m+n}\}$$

such that

$$T_f^{\mathbf{b}} = M_f^{\mathbf{b}} + \sum_{g \prec_{\mathbf{b}} f} t_{gf}^{\mathbf{b}}(q) M_g^{\mathbf{b}}, \quad L_f^{\mathbf{b}} = M_f^{\mathbf{b}} + \sum_{g \prec_{\mathbf{b}} f} \ell_{gf}^{\mathbf{b}}(q) M_g^{\mathbf{b}},$$

with $t_{gf}^{\mathbf{b}}(q) \in q\mathbb{Z}[q]$, and $\ell_{gf}^{\mathbf{b}}(q) \in q^{-1}\mathbb{Z}[q^{-1}]$, for $g \prec_{\mathbf{b}} f$. (We will also write $t_{ff}^{\mathbf{b}}(q) = \ell_{ff}^{\mathbf{b}}(q) = 1$, $t_{gf}^{\mathbf{b}} = \ell_{gf}^{\mathbf{b}} = 0$ for $g \succ_{\mathbf{b}} f$.)

Definition 3.10. $\{T_f^{\mathbf{b}} | f \in \mathbb{Z}^{m+n}\}$ and $\{L_f^{\mathbf{b}} | f \in \mathbb{Z}^{m+n}\}$ are called the *canonical basis* and *dual canonical basis* for $\widehat{\mathbb{T}}^{\mathbf{b}}$, respectively. Also, $t_{gf}^{\mathbf{b}}(q)$ and $\ell_{gf}^{\mathbf{b}}(q)$ are called *Brundan-Kazhdan-Lusztig (BKL) polynomials*.

Recall $d_i \in \mathbb{Z}^{m+n}$ from (2.5). We define

$$(3.12) \quad 1_{m|n} := \sum_{i=1}^{m+n} (-1)^{b_i} d_i \in \mathbb{Z}^{m+n}.$$

Proposition 3.11. *For each $p \in \mathbb{Z}$ and $f, g \in \mathbb{Z}^{m+n}$, we have*

$$t_{gf}^{\mathbf{b}} = t_{g+p1_{m|n}, f+p1_{m|n}}^{\mathbf{b}}, \quad \ell_{gf}^{\mathbf{b}} = \ell_{g+p1_{m|n}, f+p1_{m|n}}^{\mathbf{b}}.$$

Proof. Define a $\mathbb{Q}(q)$ -linear shift map $\mathbf{sh} : \mathbb{T}^{\mathbf{b}} \rightarrow \mathbb{T}^{\mathbf{b}}$ by

$$\mathbf{sh}(M_f^{\mathbf{b}}) := M_{f+1_{m|n}}^{\mathbf{b}}.$$

Since for $f \succeq_{\mathbf{b}} g$ if and only if $f + 1_{m|n} \succeq_{\mathbf{b}} g + 1_{m|n}$, the map \mathbf{sh} extends to a $\mathbb{Q}(q)$ -linear map on the B -completion $\widehat{\mathbb{T}}^{\mathbf{b}}$. Now \mathbf{sh} also commutes with the bar map, since the quasi- \mathcal{R} -matrix is invariant under an overall index shift by 1. Thus, we conclude that $\mathbf{sh}(T_f^{\mathbf{b}}) = T_{f+1_{m|n}}^{\mathbf{b}}$ and $\mathbf{sh}(L_f^{\mathbf{b}}) = L_{f+1_{m|n}}^{\mathbf{b}}$. The proposition follows. \square

3.5. Positivity. For $r \geq 1$ and $a \in \mathbb{Z}$, recall the divided powers $E_a^{(r)}$ and $F_a^{(r)}$. The following was conjectured in [Br1, Conjecture 2.28(iii),(iv)], in the case of the standard $0^m 1^n$ -sequence \mathbf{b}_{st} . Part (1) is a variant of [Zh, Theorem 3.3.6(3)].

Theorem 3.12. *Let \mathbf{b} be a $0^m 1^n$ -sequence and $f \in \mathbb{Z}^{m+n}$. Let $a \in \mathbb{Z}$, $r \geq 1$.*

- (1) *The elements $E_a^{(r)} T_f^{\mathbf{b}}$ and $F_a^{(r)} T_f^{\mathbf{b}}$ can be written as (possibly infinite) sums of $\{T_g^{\mathbf{b}} | g \in \mathbb{Z}^{m+n}\}$ with coefficients in $\mathbb{N}[q, q^{-1}]$.*
- (2) *The elements $E_a^{(r)} L_f^{\mathbf{b}}$ and $F_a^{(r)} L_f^{\mathbf{b}}$ can be written as (possibly infinite) sums of $\{L_g^{\mathbf{b}} | g \in \mathbb{Z}^{m+n}\}$ with coefficients in $\mathbb{N}[q, q^{-1}]$.*

Proof. For $k \in \mathbb{N}$, consider the quantum group $U(\mathfrak{gl}_{|k|})$ acting on the finite-dimensional module $\mathbb{T}_{\leq |k|}^{\mathbf{b}}$. Let us denote the canonical and dual canonical basis elements of the $U(\mathfrak{gl}_{|k|})$ -module $\mathbb{T}_{\leq |k|}^{\mathbf{b}}$ by $T_f^{(k)}$ and $L_f^{(k)}$, respectively, for $f \in \mathbb{Z}_{\leq |k|}^{m+n} := \{f \in \mathbb{Z}^{m+n} \mid |f(i)| \leq k, \forall i \in [m+n]\}$. The proof of [Lu2, Lemma 24.2.1] (cf. our Lemma 3.8) implies that the coefficients $t_{gf}^{\mathbf{b}}$ and $\ell_{gf}^{\mathbf{b}}$ are uniquely determined by the coefficients

$r_{hf}(q)$ coming from the bar-involution with $g \preceq_{\mathbf{b}} h \preceq_{\mathbf{b}} f$. Recall that such an h satisfies $|h(i)| \leq \max\{|f(j)|, |g(j)|\}$ with $j \in [m+n]$, $\forall i \in [m+n]$, by (2.6). This together with the stability (3.9) of the bar involutions for varying k implies that

$$(3.13) \quad \pi_k(T_f^{\mathbf{b}}) = T_f^{(k)} \quad \text{and} \quad \pi_k(L_f^{\mathbf{b}}) = L_f^{(k)}, \quad \forall f \in \mathbb{Z}_{\leq |k|}^{m+n}.$$

Now, we let $a \in \mathbb{Z}$ be fixed, and let $Y = T, L$. Observe that for $k > |a| + 1$ the map π_k commutes with the action of $X = E_a, F_a$. Thus, the map $X : \mathbb{T}^{\mathbf{b}} \rightarrow \mathbb{T}^{\mathbf{b}}$ given by letting X act on the left extends uniquely to a continuous map $X : \tilde{\mathbb{T}}^{\mathbf{b}} \rightarrow \tilde{\mathbb{T}}^{\mathbf{b}}$, and hence the expression $X^{(r)}Y_f^{\mathbf{b}}$ is a well-defined element in the A -completion $\tilde{\mathbb{T}}^{\mathbf{b}}$.

Let $f \in \mathbb{Z}_{\leq |k|}^{m+n}$ and choose $k > |a| + 1$. We write

$$X^{(r)}Y_f^{(k)} = \sum_{g \in \mathbb{Z}_{\leq |k|}^{m+n}} b_g^{(k)}(q)Y_g^{(k)}, \quad \text{for } b_g^{(k)}(q) \in \mathbb{Q}(q).$$

We compute, for $\ell \geq k$,

$$\pi_k \circ \pi_\ell(X^{(r)}Y_f^{\mathbf{b}}) = \pi_k(X^{(r)}Y_f^{(\ell)}) = \pi_k\left(\sum_{g \in \mathbb{Z}_{\leq |\ell|}^{m+n}} b_g^{(\ell)}(q)Y_g^{(\ell)}\right) = \sum_{g \in \mathbb{Z}_{\leq |k|}^{m+n}} b_g^{(\ell)}(q)Y_g^{(k)}.$$

On the other hand, we compute

$$\pi_k \circ \pi_\ell(X^{(r)}Y_f^{\mathbf{b}}) = \pi_k(X^{(r)}Y_f^{\mathbf{b}}) = X^{(r)}Y_f^{(k)} = \sum_{g \in \mathbb{Z}_{\leq |k|}^{m+n}} b_g^{(k)}(q)Y_g^{(k)}.$$

It follows that $b_g^{(k)}(q) = b_g^{(\ell)}(q)$, for all $g \in \mathbb{Z}_{\leq |k|}^{m+n}$ and $\ell \geq k$. Thus, we obtain

$$X^{(r)}Y_f^{\mathbf{b}} = \sum_g b_g(q)Y_g^{\mathbf{b}},$$

where $b_g(q) = b_g^{(k)}(q)$, for $g \in \mathbb{Z}_{\leq |k|}^{m+n}$. It remains to show that $b_g(q)$ lie in $\mathbb{N}[q, q^{-1}]$.

We first prove Part (1). In the case when $Y = T$, Zheng [Zh, Theorem 3.3.6(3)] proved that $b_g^{(k)}(q) \in \mathbb{N}[q, q^{-1}]$, $\forall g \in \mathbb{Z}_{\leq |k|}^{m+n}$ and $\forall k$. Thus, we conclude that in the case of $Y = T$ we have $b_g(q) \in \mathbb{N}[q, q^{-1}]$, for all $g \in \mathbb{Z}^{m+n}$, which proves (1).

Since the dual of the natural $U(\mathfrak{sl}_{|k|})$ -module is isomorphic to an exterior power of the natural module, [Br4, Theorem 11] is applicable to $\mathbb{T}_{\leq |k|}^{\mathbf{b}}$. By [Br4, Theorem 11] there exists a symmetric bilinear form $(\cdot | \cdot)$ on $\mathbb{T}_{\leq |k|}^{\mathbf{b}}$ for which the bases $\{T_f^{(k)}\}$ and $\{L_f^{(k)}\}$ are dual to each other up to some change of labeling. Furthermore, for $u, v \in \mathbb{T}_{\leq |k|}^{\mathbf{b}}$ we have $(E_a^{(r)}u | v) = (u | F_a^{(r)}v)$, for all $k > |a| + 1$, by [Br4, Lemma 3]. From this we conclude by [Zh, Theorem 3.3.6(3)] again that the positivity in (2) holds in the setting of $\mathbb{T}_{\leq |k|}^{\mathbf{b}}$ for dual canonical basis elements $L_f^{(k)}$. Now the same argument as in (1) proves (2) as well. \square

The following conjecture is a generalization of Brundan [Br1, Conjecture 2.28(i),(ii)], who conjectured it for the standard $0^m 1^n$ -sequence \mathbf{b}_{st} .

Conjecture 3.13. *Let \mathbf{b} be a $0^m 1^n$ -sequence. For $f, g \in \mathbb{Z}^{m+n}$, we have $t_{gf}^{\mathbf{b}}(q) \in \mathbb{N}[q]$, and $\ell_{gf}^{\mathbf{b}}(-q^{-1}) \in \mathbb{N}[q]$.*

4. COMPARISONS OF CANONICAL AND DUAL CANONICAL BASES

In this section, we introduce truncation maps to compare the (dual) canonical bases on Fock spaces involving $\wedge^k \mathbb{V}$ or $\wedge^k \mathbb{W}$ for varying k . We formulate a combinatorial version of super duality. A precise relationship between (dual) canonical bases of a Fock space and those of its various q -wedge subspaces is then established.

4.1. Truncation map. Let $k \in \mathbb{N} \cup \{\infty\}$. We introduce the following notations. For $f = (f_{[m+n]}, f_{[\underline{k}]}) \in \mathbb{Z}^{m+n} \times \mathbb{Z}_+^k$, set

$$M_f^{\mathbf{b},0} := M_{f_{[m+n]}}^{\mathbf{b}} \otimes \mathcal{V}_{f_{[\underline{k}]}}.$$

Then $\{M_f^{\mathbf{b},0}\}$ forms a basis, called the *standard monomial basis*, for the $\mathbb{Q}(q)$ -vector space $\mathbb{T}^{\mathbf{b}} \otimes \wedge^k \mathbb{V}$. Similarly, $\mathbb{T}^{\mathbf{b}} \otimes \wedge^k \mathbb{W}$ admits a *standard monomial basis* given by

$$M_g^{\mathbf{b},1} := M_{g_{[m+n]}}^{\mathbf{b}} \otimes \mathcal{W}_{g_{[\underline{k}]}}.$$

where $g = (g_{[m+n]}, g_{[\underline{k}]}) \in \mathbb{Z}^{m+n} \times \mathbb{Z}_-^k$. Since $\wedge^k \mathbb{V}$ and $\wedge^k \mathbb{W}$ are weakly based modules, we can define bar maps for $\mathbb{T}^{\mathbf{b}} \otimes \wedge^k \mathbb{V}$ and $\mathbb{T}^{\mathbf{b}} \otimes \wedge^k \mathbb{W}$ by means of the quasi- \mathcal{R} -matrix as in (3.8) and (3.10). In this subsection we prove the existence of canonical and dual canonical bases in the B -completions of these vector spaces. We shall give the details only for $\mathbb{T}^{\mathbf{b}} \otimes \wedge^k \mathbb{W}$, as the case of $\mathbb{T}^{\mathbf{b}} \otimes \wedge^k \mathbb{V}$ is analogous.

First suppose that $k \in \mathbb{N}$. Since H_0 is bar-invariant by (2.8), we may embed $\wedge^k \mathbb{W}$ into $\mathbb{W}^{\otimes k}$ as weakly based modules by sending \mathcal{W}_h to $M_{h \cdot w_0^{(k)}}^{(1^k)} H_0$, for $h \in \mathbb{Z}_-^k$. Thus, we have $\mathbb{T}^{\mathbf{b}} \otimes \wedge^k \mathbb{W} \subseteq \mathbb{T}^{\mathbf{b}} \otimes \mathbb{W}^{\otimes k}$, and hence $\mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^k \mathbb{W} \subseteq \mathbb{T}^{\mathbf{b}} \widehat{\otimes} \mathbb{W}^{\otimes k} \equiv \widehat{\mathbb{T}}^{(\mathbf{b}, 1^k)}$, where $\mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^k \mathbb{W}$ is a similarly defined B -completion with respect to the Bruhat ordering of type $(\mathbf{b}, 1^k)$, following Definition 3.2. Expanding $M_{h \cdot w_0^{(k)}}^{(1^k)} H_0$ in terms of the $M_e^{(1^k)}$'s, and using Propositions 3.5 and 3.6, we conclude that

$$(4.1) \quad \overline{M_f^{\mathbf{b},1}} = M_f^{\mathbf{b},1} + \sum_{g \prec_{(\mathbf{b}, 1^k)} f} r_{gf}(q) M_g^{\mathbf{b},1},$$

where $r_{gf}(q) \in \mathbb{Z}[q, q^{-1}]$, and the sum running over $g \in \mathbb{Z}^{m+n} \times \mathbb{Z}_-^k$ is possibly infinite. It follows that we have obtained a bar-involution $\bar{\cdot} : \mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^k \mathbb{W} \rightarrow \mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^k \mathbb{W}$, exactly as in Lemma 3.7.

Now let $k = \infty$. For $d \in \mathbb{N}$, let $[\mathbb{T}^{\mathbf{b}} \otimes \wedge^\infty \mathbb{W}]_{\leq |d|}$ be the subspace of $\mathbb{T}^{\mathbf{b}} \otimes \wedge^\infty \mathbb{W}$ spanned by vectors $M_f^{\mathbf{b},1}$, for $f \in \mathbb{Z}^{m+n} \times \mathbb{Z}_-^\infty$ with $|f(i)| \leq d$, for $i \in [m+n] \cup [d]$. We let

$$\pi'_d : \mathbb{T}^{\mathbf{b}} \otimes \wedge^\infty \mathbb{W} \rightarrow [\mathbb{T}^{\mathbf{b}} \otimes \wedge^\infty \mathbb{W}]_{\leq |d|}$$

be the natural projection map. Then we may use the $\ker \pi'_d$'s to define the A -completion $\mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^\infty \mathbb{W}$ of $\mathbb{T}^{\mathbf{b}} \otimes \wedge^\infty \mathbb{W}$, following Definition 3.2.

Let $M_f^{\mathbf{b},1} \in [\mathbb{T}^{\mathbf{b}} \otimes \wedge^\infty \mathbb{W}]_{\leq |d|}$. Using (3.9) and an argument similar to its proof we can show that, for $\ell \geq d$,

$$(4.2) \quad \pi'_d(\psi^{(\ell)}(M_f^{\mathbf{b},1})) = \pi'_d(\psi^{(d)}(M_f^{\mathbf{b},1})).$$

It follows that the expression $\overline{M_f^{\mathbf{b},1}}$, defined as in (3.8) and (3.10) on the tensor product of the two weakly based modules $\mathbb{T}^{\mathbf{b}}$ and $\wedge^\infty \mathbb{W}$, lies in $\mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^\infty \mathbb{W}$. Let $\mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^\infty \mathbb{W}$ be the B -completion of $\mathbb{T}^{\mathbf{b}} \otimes \wedge^\infty \mathbb{W}$, following Definition 3.2.

For $h \in \mathbb{Z}_-^\infty$ and $k \in \mathbb{N}$ recall that $h_{[k]}$ denotes the restriction of h to $[k]$. We define the $\mathbb{Q}(q)$ -linear *truncation map* $\text{Tr} : \mathbb{T}^{\mathbf{b}} \otimes \wedge^\infty \mathbb{W} \rightarrow \mathbb{T}^{\mathbf{b}} \otimes \wedge^k \mathbb{W}$, for $k \in \mathbb{N}$, as follows. For $m \in \mathbb{T}^{\mathbf{b}}$ and $h \in \mathbb{Z}_-^\infty$ we set

$$\text{Tr}(m \otimes \mathcal{W}_h) = \begin{cases} m \otimes \mathcal{W}_{h_{[k]}}, & \text{if } h(\underline{i}) = i, \text{ for } i \geq k+1, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 4.1. *Let $k \in \mathbb{N}$. The truncation map extends naturally to a $\mathbb{Q}(q)$ -linear map $\text{Tr} : \mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^\infty \mathbb{W} \rightarrow \mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^k \mathbb{W}$.*

Proof. By definition of the B -completions, it is enough to prove that the two Bruhat orderings on $\mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^\infty \mathbb{W}$ and $\mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^k \mathbb{W}$ are compatible under the truncation map.

Let $f, g \in \mathbb{Z}^{m+n} \times \mathbb{Z}_-^\infty$ with $f \succ_{(\mathbf{b},1^\infty)} g$. Suppose first that $\text{Tr}(M_f^{\mathbf{b},1}) = M_{f'}^{\mathbf{b},1} \neq 0$ and $\text{Tr}(M_g^{\mathbf{b},1}) = M_{g'}^{\mathbf{b},1} \neq 0$. Then we have $f_{[m+n] \sqcup [k]} = f'$, $g_{[m+n] \sqcup [k]} = g'$, and $f(\underline{i}) = g(\underline{i}) = i$, for all $i \geq k+1$. It follows from the very definition of the Bruhat ordering that $f' \succ_{(\mathbf{b},1^k)} g'$.

Now suppose that $\text{Tr}(M_f^{\mathbf{b},1}) = 0$ and $f \succ_{(\mathbf{b},1^\infty)} g$. If $f_{[\infty]} = g_{[\infty]}$, then clearly $\text{Tr}(M_g^{\mathbf{b},1}) = 0$. If not, then let \underline{i} with i minimal so that $f(\underline{i}) \neq g(\underline{i})$. If $i \leq k$, then again $\text{Tr}(M_g^{\mathbf{b},1}) = 0$. So suppose that $i > k$. Since $f \succ_{(\mathbf{b},1^\infty)} g$, we must have $g(\underline{i}) < f(\underline{i}) \leq i$, and so again we have $\text{Tr}(M_g^{\mathbf{b},1}) = 0$. Thus, we have shown that $\text{Tr}(M_f^{\mathbf{b},1}) = 0$ implies that $\text{Tr}(M_g^{\mathbf{b},1}) = 0$. \square

Lemma 4.2. *$\text{Tr} : \mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^\infty \mathbb{W} \rightarrow \mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^k \mathbb{W}$ commutes with the bar maps, that is,*

$$\overline{\text{Tr}(M_f^{\mathbf{b},1})} = \text{Tr}(\overline{M_f^{\mathbf{b},1}}), \text{ for } f \in \mathbb{Z}^{m+n} \times \mathbb{Z}_-^\infty.$$

Moreover, we have

$$(4.3) \quad \overline{M_f^{\mathbf{b},1}} = M_f^{\mathbf{b},1} + \sum_{g \prec_{(\mathbf{b},1^\infty)} f} r_{gf}(q) M_g^{\mathbf{b},1}, \text{ for } r_{gf}(q) \in \mathbb{Z}[q, q^{-1}].$$

Proof. Recall from (3.8) and (3.10) that the bar maps on the tensor spaces $\mathbb{T}^{\mathbf{b}} \otimes \wedge^\infty \mathbb{W}$ and $\mathbb{T}^{\mathbf{b}} \otimes \wedge^k \mathbb{W}$ are defined by the formula $\overline{m \otimes n} = \Theta(\overline{m} \otimes \overline{n})$, where $\Theta = \sum_{\mu \in \sum \mathbb{Z}_+ \oplus \Phi^+} \Theta_\mu \otimes \Theta_{-\mu}$ is given in (3.4) with $\Theta_{-\mu} \in \mathcal{U}^-$. It is easily verified that the map $\text{Tr}_> : \wedge^\infty \mathbb{W} \rightarrow \wedge^k \mathbb{W}$ defined by

$$\text{Tr}_>(\mathcal{W}_h) = \begin{cases} \mathcal{W}_{h_{[k]}}, & \text{if } h(\underline{i}) = i \text{ (for } i \geq k+1), \\ 0, & \text{otherwise,} \end{cases}, \quad \text{for } h \in \mathbb{Z}_-^\infty,$$

is a \mathcal{U}^- -module homomorphism. Therefore we have

$$\sum_{\mu} \mathrm{Tr} [\Theta_{\mu} \overline{m} \otimes \Theta_{-\mu} (w_{h(\underline{1})} \wedge w_{h(\underline{2})} \wedge \cdots)] = \sum_{\mu} \Theta_{\mu} \overline{m} \otimes \Theta_{-\mu} (w_{h(\underline{1})} \wedge \cdots \wedge w_{h(\underline{k})}).$$

From this, it follows that $\mathrm{Tr} : \mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^{\infty} \mathbb{W} \rightarrow \mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^k \mathbb{W}$ commutes with the bar maps. Now (4.3) follows from (4.1). \square

From Lemma 4.2 and the bar-involutions on $\mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^k \mathbb{W}$ for $k \in \mathbb{N}$, we obtain a bar-involution $\bar{\cdot} : \mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^{\infty} \mathbb{W} \rightarrow \mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^{\infty} \mathbb{W}$. Summarizing the above, and applying Lemmas 3.8 and 4.2 we have proved the following.

Proposition 4.3. *Let $k \in \mathbb{N} \cup \{\infty\}$. The bar map $\bar{\cdot} : \mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^k \mathbb{W} \rightarrow \mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^k \mathbb{W}$ is an involution. The space $\mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^k \mathbb{W}$ has unique bar-invariant topological bases*

$$\{T_f^{\mathbf{b},1} | f \in \mathbb{Z}^{m+n} \times \mathbb{Z}_-^k\} \text{ and } \{L_f^{\mathbf{b},1} | f \in \mathbb{Z}^{m+n} \times \mathbb{Z}_-^k\}$$

such that

$$T_f^{\mathbf{b},1} = M_f^{\mathbf{b},1} + \sum_{g \prec_{(\mathbf{b},1^k)} f} t_{gf}^{\mathbf{b},1}(q) M_g^{\mathbf{b},1}, \quad L_f^{\mathbf{b},1} = M_f^{\mathbf{b},1} + \sum_{g \prec_{(\mathbf{b},1^k)} f} \ell_{gf}^{\mathbf{b},1}(q) M_g^{\mathbf{b},1},$$

with $t_{gf}^{\mathbf{b},1}(q) \in q\mathbb{Z}[q]$, and $\ell_{gf}^{\mathbf{b},1}(q) \in q^{-1}\mathbb{Z}[q^{-1}]$. (We will write $t_{ff}^{\mathbf{b},1}(q) = \ell_{ff}^{\mathbf{b},1}(q) = 1$, $t_{gf}^{\mathbf{b},1} = \ell_{gf}^{\mathbf{b},1} = 0$, for $g \not\prec_{(\mathbf{b},1^k)} f$.)

We call $\{T_f^{\mathbf{b},1}\}$ and $\{L_f^{\mathbf{b},1}\}$ the *canonical and dual canonical bases* of $\mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^k \mathbb{W}$, respectively. We shall use $f^{\underline{k}} \in \mathbb{Z}^{m+n} \times \mathbb{Z}_-^k$ as a short-hand notation for the restriction $f|_{[m+n] \sqcup [\underline{k}]}$ of a function $f \in \mathbb{Z}^{m+n} \times \mathbb{Z}_-^{\infty}$.

Proposition 4.4. *Let $k \in \mathbb{N}$. The truncation map $\mathrm{Tr} : \mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^{\infty} \mathbb{W} \rightarrow \mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^k \mathbb{W}$ preserves the standard, canonical, and dual canonical bases in the following sense: for $Y = M, L, T$ and $f \in \mathbb{Z}^{m+n} \times \mathbb{Z}_-^{\infty}$ we have*

$$\mathrm{Tr} \left(Y_f^{\mathbf{b},1} \right) = \begin{cases} Y_{f^{\underline{k}}}^{\mathbf{b},1}, & \text{if } f(\underline{i}) = i, \text{ for } i \geq k+1, \\ 0, & \text{otherwise.} \end{cases}$$

Consequently, we have $t_{gf}^{\mathbf{b},1}(q) = t_{g^{\underline{k}}f^{\underline{k}}}^{\mathbf{b},1}(q)$ and $\ell_{gf}^{\mathbf{b},1}(q) = \ell_{g^{\underline{k}}f^{\underline{k}}}^{\mathbf{b},1}(q)$, for $g, f \in \mathbb{Z}^{m+n} \times \mathbb{Z}_-^{\infty}$ such that $f(\underline{i}) = g(\underline{i}) = i$, for $i \geq k+1$.

Proof. By definition the statement is true for $Y = M$. By Lemmas 4.1 and 4.2 the map $\mathrm{Tr} : \mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^{\infty} \mathbb{W} \rightarrow \mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^k \mathbb{W}$ is compatible with canonical and dual canonical bases of these two spaces. \square

The constructions and statements when replacing $\wedge^k \mathbb{W}$ by $\wedge^k \mathbb{V}$, for $k \in \mathbb{N} \cup \{\infty\}$ are entirely analogous, and so we will skip the analogous proofs. We construct a B -completion $\mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^k \mathbb{V}$. For $k \in \mathbb{N}$, the truncation map $\mathrm{Tr} : \mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^{\infty} \mathbb{V} \rightarrow \mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^k \mathbb{V}$, is defined by

$$\mathrm{Tr}(m \otimes \mathcal{V}_h) = \begin{cases} m \otimes \mathcal{V}_{h|_{[\underline{k}]}} & \text{if } h(\underline{i}) = 1 - i, \text{ for } i \geq k+1, \\ 0, & \text{otherwise,} \end{cases}$$

where $m \in \mathbb{T}^{\mathbf{b}}$ and $h \in \mathbb{Z}_+^\infty$. The map \mathbf{Tr} extends to the B -completions. The following is a $\wedge^k \mathbb{V}$ -analogue of Proposition 4.3.

Proposition 4.5. *Let $k \in \mathbb{N} \cup \{\infty\}$. We have a bar-involution $\bar{\cdot} : \mathbb{T}^{\mathbf{b}} \hat{\otimes} \wedge^k \mathbb{V} \rightarrow \mathbb{T}^{\mathbf{b}} \hat{\otimes} \wedge^k \mathbb{V}$. The space $\mathbb{T}^{\mathbf{b}} \hat{\otimes} \wedge^k \mathbb{V}$ has unique bar-invariant topological bases*

$$\{T_f^{\mathbf{b},0} | f \in \mathbb{Z}^{m+n} \times \mathbb{Z}_+^k\} \text{ and } \{L_f^{\mathbf{b},0} | f \in \mathbb{Z}^{m+n} \times \mathbb{Z}_+^k\}$$

such that

$$T_f^{\mathbf{b},0} = M_f^{\mathbf{b},0} + \sum_{g \prec_{(\mathbf{b},0^k)} f} t_{gf}^{\mathbf{b},0}(q) M_g^{\mathbf{b},0}, \quad L_f^{\mathbf{b},0} = M_f^{\mathbf{b},0} + \sum_{g \prec_{(\mathbf{b},0^k)} f} \ell_{gf}^{\mathbf{b},0}(q) M_g^{\mathbf{b},0},$$

with $t_{gf}^{\mathbf{b},0}(q) \in q\mathbb{Z}[q]$, and $\ell_{gf}^{\mathbf{b},0}(q) \in q^{-1}\mathbb{Z}[q^{-1}]$. (We will write $t_{ff}^{\mathbf{b},0}(q) = \ell_{ff}^{\mathbf{b},0}(q) = 1$, $t_{gf}^{\mathbf{b},0} = \ell_{gf}^{\mathbf{b},0} = 0$, for $g \not\prec_{(\mathbf{b},0^k)} f$.)

We call $\{T_f^{\mathbf{b},0}\}$ and $\{L_f^{\mathbf{b},0}\}$ the *canonical* and *dual canonical* bases of $\mathbb{T}^{\mathbf{b}} \hat{\otimes} \wedge^k \mathbb{V}$, respectively.

We shall also use $f_{\underline{k}} \in \mathbb{Z}^{m+n} \times \mathbb{Z}_+^k$ as a short-hand notation for the restriction $f_{[m+n] \sqcup [\underline{k}]}$ of a function $f \in \mathbb{Z}^{m+n} \times \mathbb{Z}_+^\infty$. The following is a $\wedge^k \mathbb{V}$ -analogue of Lemma 4.2 and Proposition 4.4.

Proposition 4.6. *The truncation map $\mathbf{Tr} : \mathbb{T}^{\mathbf{b}} \hat{\otimes} \wedge^\infty \mathbb{V} \rightarrow \mathbb{T}^{\mathbf{b}} \hat{\otimes} \wedge^k \mathbb{V}$ commutes with the bar involutions. Moreover, the truncation map \mathbf{Tr} preserves the standard, canonical, and dual canonical bases; that is, for $Y = M, L, T$, and for $f \in \mathbb{Z}^{m+n} \times \mathbb{Z}_+^\infty$, we have*

$$\mathbf{Tr}(Y_f^{\mathbf{b},0}) = \begin{cases} Y_{f_{\underline{k}}}^{\mathbf{b},0}, & \text{if } f(\underline{i}) = 1 - i, \text{ for } i \geq k+1, \\ 0, & \text{otherwise.} \end{cases}$$

Consequently, we have $t_{gf}^{\mathbf{b},0}(q) = t_{g_{\underline{k}}f_{\underline{k}}}^{\mathbf{b},0}(q)$ and $\ell_{gf}^{\mathbf{b},0}(q) = \ell_{g_{\underline{k}}f_{\underline{k}}}^{\mathbf{b},0}(q)$, for $g, f \in \mathbb{Z}^{m+n} \times \mathbb{Z}_+^\infty$ such that $f(\underline{i}) = g(\underline{i}) = 1 - i$, for $i \geq k+1$.

Definition 4.7. The polynomials $\ell_{gf}^{\mathbf{b},0}(q)$, $\ell_{gf}^{\mathbf{b},1}(q)$, $t_{gf}^{\mathbf{b},0}(q)$, $t_{gf}^{\mathbf{b},1}(q)$ are called *Brundan-Kazhdan-Lusztig (BKL) polynomials*. (They include $\ell_{gf}^{\mathbf{b}}(q)$ and $t_{gf}^{\mathbf{b}}(q)$ in Definition 3.10 as special cases).

Note that the BKL polynomials reduce to the usual (parabolic) Kazhdan-Lusztig polynomials when the underlying Fock spaces involve only \mathbb{V} (or only \mathbb{W}).

4.2. Combinatorial super duality. Recall from Proposition 2.7 the isomorphism of $U_q(\mathfrak{sl}_\infty)$ -modules $\mathfrak{h} : \wedge^\infty \mathbb{V} \rightarrow \wedge^\infty \mathbb{W}$ defined by $\mathfrak{h}(|\lambda\rangle) = |\lambda'_\star\rangle$. This map induces an isomorphism

$$\mathfrak{h}_{\mathbf{b}} \stackrel{\text{def}}{=} 1_{\mathbf{b}} \otimes \mathfrak{h} : \mathbb{T}^{\mathbf{b}} \otimes \wedge^\infty \mathbb{V} \longrightarrow \mathbb{T}^{\mathbf{b}} \otimes \wedge^\infty \mathbb{W},$$

where $1_{\mathbf{b}}$ denotes the identity map on $\mathbb{T}^{\mathbf{b}}$. Let $f \in \mathbb{Z}^{m+n} \times \mathbb{Z}_+^\infty$. There exists a unique $\lambda \in \mathcal{P}$ such that $|\lambda\rangle = \mathcal{V}_{f_{[\underline{\infty}]}}$. We define $f^{\mathfrak{h}}$ to be the unique element in $\mathbb{Z}^{m+n} \times \mathbb{Z}_-^\infty$

determined by $f^{\natural}(i) = f(i)$, for $i \in [m+n]$, and $\mathcal{W}_{f^{\natural}[\infty]} = |\lambda'_*|$. The assignment $f \mapsto f^{\natural}$ gives a bijection (cf. [CWZ])

$$(4.4) \quad \natural : \mathbb{Z}^{m+n} \times \mathbb{Z}_+^{\infty} \longrightarrow \mathbb{Z}^{m+n} \times \mathbb{Z}_-^{\infty}.$$

The following is the combinatorial counterpart of the super duality in Theorem 7.2 later on representation theory.

- Theorem 4.8.** (1) *The isomorphism $\natural_{\mathbf{b}}$ respects the Bruhat orderings, and hence extends to an isomorphism of the B -completions $\natural_{\mathbf{b}} : \mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^{\infty} \mathbb{V} \rightarrow \mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^{\infty} \mathbb{W}$.*
- (2) *$\natural_{\mathbf{b}}$ commutes with the bar involutions.*
- (3) *$\natural_{\mathbf{b}}$ preserves the canonical and dual canonical bases. More precisely, we have $\natural_{\mathbf{b}}(M_f^{\mathbf{b},0}) = M_{f^{\natural}}^{\mathbf{b},1}$, $\natural_{\mathbf{b}}(T_f^{\mathbf{b},0}) = T_{f^{\natural}}^{\mathbf{b},1}$ and $\natural_{\mathbf{b}}(L_f^{\mathbf{b},0}) = L_{f^{\natural}}^{\mathbf{b},1}$, for $f \in \mathbb{Z}^{m+n} \times \mathbb{Z}_+^{\infty}$.*
- (4) *We have the following identifications of BKL polynomials: $\ell_{gf}^{\mathbf{b},0}(q) = \ell_{g^{\natural}f^{\natural}}^{\mathbf{b},1}(q)$, and $t_{gf}^{\mathbf{b},0}(q) = t_{g^{\natural}f^{\natural}}^{\mathbf{b},1}(q)$, for all $g, f \in \mathbb{Z}^{m+n} \times \mathbb{Z}_+^{\infty}$.*

Proof. We first prove (2)–(4) by assuming that (1) holds. Since $\wedge^{\infty} \mathbb{V}$ and $\wedge^{\infty} \mathbb{W}$ are isomorphic $U_q(\mathfrak{sl}_{\infty})$ -modules under \natural , $\natural_{\mathbf{b}}$ is compatible with the quasi- \mathcal{R} -matrices. Thus, $\natural_{\mathbf{b}}$ commutes with the bar involutions (see (3.8)), whence (2). It follows by definition that $\natural_{\mathbf{b}}(M_f^{\mathbf{b},0}) = M_{f^{\natural}}^{\mathbf{b},1}$. The commutativity of $\natural_{\mathbf{b}}$ with the bar-involutions implies that canonical and dual canonical bases are sent by $\natural_{\mathbf{b}}$ to bar invariant elements. Now by the compatibility of the two Bruhat orderings in (1), $\natural_{\mathbf{b}}(T_f^{\mathbf{b},0})$ and $\natural_{\mathbf{b}}(L_f^{\mathbf{b},0})$ satisfy the same characterization properties as the canonical and dual canonical basis elements $T_{f^{\natural}}^{\mathbf{b},1}$ and $L_{f^{\natural}}^{\mathbf{b},1}$ respectively, and hence (3) follows. Now (3) clearly implies (4).

It remains to show (1). To this end, let $f, g \in \mathbb{Z}^{m+n} \times \mathbb{Z}_+^{\infty}$. We shall show that $f \succeq_{(\mathbf{b},0^{\infty})} g$ is equivalent to $f^{\natural} \succeq_{(\mathbf{b},1^{\infty})} g^{\natural}$. Recall that we denote the restriction of f to a subset I by f_I .

[CWZ, Lemma 6.2] says that

$$(4.5) \quad \{f(\underline{j}) | j \in \mathbb{N}\} \sqcup \{f^{\natural}(\underline{j}) | j \in \mathbb{N}\} = \mathbb{Z},$$

(and similar claim when replacing f by g). Choose $N \gg 0$ so that $f(\underline{t}) = g(\underline{t}) = -t+1$, $f^{\natural}(\underline{t}) = g^{\natural}(\underline{t}) = t$, for all $t > N$. By our choice of N and (4.5), we have

$$\begin{aligned} \{f(\underline{j}) | 1 \leq j \leq N\} \sqcup \{f^{\natural}(\underline{j}) | 1 \leq j \leq N\} &= \{g(\underline{j}) | 1 \leq j \leq N\} \sqcup \{g^{\natural}(\underline{j}) | 1 \leq j \leq N\} \\ &= \{-N+1, -N+2, \dots, N-1, N\}. \end{aligned}$$

From this we conclude that, for $-N+1 \leq d \leq N$,

$$(4.6) \quad \sharp_{(0^N)}(f_{[\underline{N}]}, d, \underline{1}) - \sharp_{(1^N)}(f_{[\underline{N}]}^{\natural}, d, \underline{1}) = N + d = \sharp_{(0^N)}(g_{[\underline{N}]}, d, \underline{1}) - \sharp_{(1^N)}(g_{[\underline{N}]}^{\natural}, d, \underline{1}).$$

Here we recall the notation $\sharp_{\mathbf{b}}$ from (2.3) and that a Bruhat ordering $\succeq_{\mathbf{b}}$ is characterized in terms of $\sharp_{\mathbf{b}}$ by (2.4).

The condition $f \succeq_{(\mathbf{b},0^{\infty})} g$ is equivalent to $f_{[m+n] \cup [\underline{N}]} \succeq_{(\mathbf{b},0^N)} g_{[m+n] \cup [\underline{N}]}$, which is equivalent by (2.4) to

$$(4.7) \quad \sharp_{(\mathbf{b},0^N)}(f_{[m+n] \cup [\underline{N}]}, d, j) \geq \sharp_{(\mathbf{b},0^N)}(g_{[m+n] \cup [\underline{N}]}, d, j), \quad \forall j \in [m+n] \cup [\underline{N}], \forall d \in \mathbb{Z},$$

with equality holding for $j = 1$.

On the other hand, $f^{\natural} \succeq_{(\mathbf{b}, 1^\infty)} g^{\natural}$ is equivalent to $f^{\natural}_{[m+n] \cup [N]} \succeq_{(\mathbf{b}, 1^N)} g^{\natural}_{[m+n] \cup [N]}$, which in turn, via (2.4), is equivalent to

$$(4.8) \quad \sharp_{(\mathbf{b}, 1^N)}(f^{\natural}_{[m+n] \cup [N]}, d, j) \geq \sharp_{(\mathbf{b}, 1^N)}(g^{\natural}_{[m+n] \cup [N]}, d, j), \quad \forall j \in [m+n] \cup [N], \forall d \in \mathbb{Z},$$

with equality holding for $j = 1$.

Thus, we are reduced to show the equivalence between (4.7) and (4.8). We separate into several cases below.

(a) Consider the case for $j \in [N]$ (and d arbitrary). Denote by $\lambda = (\lambda_1, \lambda_2, \dots)$ the partition defined by letting $\lambda_i = f(\underline{i}) + i - 1$, and by $\mu = (\mu_1, \mu_2, \dots)$ the partition defined by letting $\mu_i = g(\underline{i}) + i - 1$, for $i \geq 1$. Then (4.7) in the case for $j \in [N]$ is equivalent to $\lambda \subseteq \mu$. This is equivalent to $\lambda' \subseteq \mu'$, which in turn is equivalent to (4.8) in the case for $j \in [N]$.

(b) Consider the case for $d < -N + 1$ and $j \in [m+n]$. Then $\sharp_{(\mathbf{b}, 0^N)}(f_{[m+n] \cup [N]}, d, j) = \sharp_{(\mathbf{b}, 1^N)}(f^{\natural}_{[m+n] \cup [N]}, d, j)$, and similarly, $\sharp_{(\mathbf{b}, 0^N)}(g_{[m+n] \cup [N]}, d, j) = \sharp_{(\mathbf{b}, 1^N)}(g^{\natural}_{[m+n] \cup [N]}, d, j)$. Thus, it follows that (4.7) and (4.8) are equivalent in this case.

(c) Now consider the case for $d > N$ and $j \in [m+n]$. We have

$$\begin{aligned} \sharp_{(\mathbf{b}, 0^N)}(f_{[m+n] \cup [N]}, d, j) &= \sharp_{\mathbf{b}}(f_{[m+n]}, d, j) + N, \\ \sharp_{(\mathbf{b}, 1^N)}(f^{\natural}_{[m+n] \cup [N]}, d, j) &= \sharp_{\mathbf{b}}(f_{[m+n]}, d, j) - N, \\ \sharp_{(\mathbf{b}, 0^N)}(g_{[m+n] \cup [N]}, d, j) &= \sharp_{\mathbf{b}}(g_{[m+n]}, d, j) + N, \\ \sharp_{(\mathbf{b}, 1^N)}(g^{\natural}_{[m+n] \cup [N]}, d, j) &= \sharp_{\mathbf{b}}(g_{[m+n]}, d, j) - N. \end{aligned}$$

Thus, (4.7) and (4.8) are equivalent in this case as well.

(d) Finally, consider the case for $-N + 1 \leq d \leq N$ and $j \in [m+n]$. We have the following equivalent statements:

$$\begin{aligned} \sharp_{(\mathbf{b}, 0^N)}(f_{[m+n] \cup [N]}, d, j) &\geq \sharp_{(\mathbf{b}, 0^N)}(g_{[m+n] \cup [N]}, d, j) \\ \iff \sharp_{\mathbf{b}}(f_{[m+n]}, d, j) + \sharp_{(0^N)}(f_{[N]}, d, \underline{1}) &\geq \sharp_{\mathbf{b}}(g_{[m+n]}, d, j) + \sharp_{(0^N)}(g_{[N]}, d, \underline{1}) \\ \stackrel{(4.6)}{\iff} \sharp_{\mathbf{b}}(f_{[m+n]}, d, j) + \sharp_{(1^N)}(f^{\natural}_{[N]}, d, \underline{1}) &\geq \sharp_{\mathbf{b}}(g_{[m+n]}, d, j) + \sharp_{(1^N)}(g^{\natural}_{[N]}, d, \underline{1}) \\ \iff \sharp_{(\mathbf{b}, 1^N)}(f^{\natural}_{[m+n] \cup [N]}, d, j) &\geq \sharp_{(\mathbf{b}, 1^N)}(g^{\natural}_{[m+n] \cup [N]}, d, j). \end{aligned}$$

We observe that in (b)–(d) when $j = 1$ we have equality in (4.7) if and only if we have equality in (4.8). Summarizing (a)–(d), we have shown the equivalence of (4.7) and (4.8), and hence, by (2.4), the equivalence between $f \succeq_{(\mathbf{b}, 0^\infty)} g$ and $f^{\natural} \succeq_{(\mathbf{b}, 1^\infty)} g^{\natural}$.

The compatibility of $\natural_{\mathbf{b}}$ with the two Bruhat orderings implies that $\natural_{\mathbf{b}}$ extends to the respective B -completions, $\natural_{\mathbf{b}} : \mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^\infty \mathbb{V} \longrightarrow \mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^\infty \mathbb{W}$. This completes the proof of (1) and hence the proof of the theorem. \square

4.3. Tensor versus q -wedges. Let \mathbf{b} be a fixed $0^m 1^n$ -sequence and let $k \in \mathbb{N}$. We shall compare canonical and dual canonical bases of the space $\mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^k \mathbb{V}$ (respectively, $\mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^k \mathbb{W}$) with those of $\mathbb{T}^{\mathbf{b}} \widehat{\otimes} \mathbb{V}^{\otimes k}$ (respectively, $\mathbb{T}^{\mathbf{b}} \widehat{\otimes} \mathbb{W}^{\otimes k}$). We shall only do this comparison for $\mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^k \mathbb{V}$ and $\mathbb{T}^{\mathbf{b}} \widehat{\otimes} \mathbb{V}^{\otimes k}$ in detail, the other case being analogous.

For $k \in \mathbb{N}$, recall that we may regard $\mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^k \mathbb{V} \subseteq \mathbb{T}^{\mathbf{b}} \widehat{\otimes} \mathbb{V}^{\otimes k}$ with compatible bar involutions via the identification \mathcal{V}_h with $M_{h \cdot w_0^{(k)}}^{(0^k)} H_0$, for $h \in \mathbb{Z}_+^k$.

Let $f \in \mathbb{Z}^{m+n} \times \mathbb{Z}_+^k$. As before, we write the dual canonical basis element $L_f^{(\mathbf{b}, 0^k)}$ in $\mathbb{T}^{\mathbf{b}} \widehat{\otimes} \mathbb{V}^{\otimes k}$ and the corresponding dual canonical basis element $L_f^{\mathbf{b}, 0}$ in $\mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^k \mathbb{V}$ as

$$(4.9) \quad L_f^{(\mathbf{b}, 0^k)} = \sum_{g \in \mathbb{Z}^{m+n} \times \mathbb{Z}^k} \ell_{gf}^{(\mathbf{b}, 0^k)}(q) M_g^{(\mathbf{b}, 0^k)},$$

$$(4.10) \quad L_f^{\mathbf{b}, 0} = \sum_{g \in \mathbb{Z}^{m+n} \times \mathbb{Z}_+^k} \ell_{gf}^{\mathbf{b}, 0}(q) M_g^{\mathbf{b}, 0}.$$

The following proposition states that the BKL-polynomials ℓ 's in $\mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^k \mathbb{V}$ coincide with their counterparts in $\mathbb{T}^{\mathbf{b}} \widehat{\otimes} \mathbb{V}^{\otimes k}$.

Proposition 4.9. *Let $f, g \in \mathbb{Z}^{m+n} \times \mathbb{Z}_+^k$. Then $\ell_{gf}^{\mathbf{b}, 0}(q) = \ell_{gf}^{(\mathbf{b}, 0^k)}(q)$.*

Proof. The argument below is adapted from the one given in [Br1, Page 205].

Via the identification $\mathcal{V}_{g_{[\underline{k}]}} \equiv M_{g_{[\underline{k}]} \cdot w_0^{(k)}}^{(0^k)} H_0$, (2.7) and (2.9), we write $M_g^{\mathbf{b}, 0} = M_g^{(\mathbf{b}, 0^k)} + (*)$, where $(*)$ is a $q^{-1}\mathbb{Z}[q^{-1}]$ -linear combination of $M_h^{(\mathbf{b}, 0^k)}$, with h satisfying $h \prec_{(\mathbf{b}, 0^k)} g$ and $h \notin \mathbb{Z}^{m+n} \times \mathbb{Z}_+^k$. When combining this with (4.10), we can write $L_f^{\mathbf{b}, 0} = M_f^{(\mathbf{b}, 0^k)} + (**)$, where $(**)$ is a $q^{-1}\mathbb{Z}[q^{-1}]$ -linear combination of $M_g^{(\mathbf{b}, 0^k)}$, with g satisfying $g \prec_{(\mathbf{b}, 0^k)} f$. Since $L_f^{\mathbf{b}, 0}$ is also bar-invariant, this expression equals $L_f^{(\mathbf{b}, 0^k)}$, by the uniqueness of the dual canonical basis. The proposition now follows by comparing the coefficients of $M_g^{(\mathbf{b}, 0^k)}$, for $g \in \mathbb{Z}^{m+n} \times \mathbb{Z}_+^k$, in this expression and in (4.9). \square

Let $f \in \mathbb{Z}^{m+n} \times \mathbb{Z}_+^k$. Similarly as before we write the canonical basis element $T_f^{(\mathbf{b}, 0^k)}$ in $\mathbb{T}^{\mathbf{b}} \widehat{\otimes} \mathbb{V}^{\otimes k}$ and the canonical basis element $T_f^{\mathbf{b}, 0}$ in $\mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^k \mathbb{V}$ respectively as

$$(4.11) \quad T_f^{(\mathbf{b}, 0^k)} = \sum_{g \in \mathbb{Z}^{m+n} \times \mathbb{Z}^k} t_{gf}^{(\mathbf{b}, 0^k)}(q) M_g^{(\mathbf{b}, 0^k)},$$

$$(4.12) \quad T_f^{\mathbf{b}, 0} = \sum_{g \in \mathbb{Z}^{m+n} \times \mathbb{Z}_+^k} t_{gf}^{\mathbf{b}, 0}(q) M_g^{\mathbf{b}, 0}.$$

Recall $w_0^{(k)}$ is the longest element in \mathfrak{S}_k .

Proposition 4.10. *For $f, g \in \mathbb{Z}^{m+n} \times \mathbb{Z}_+^k$, we have*

$$t_{gf}^{\mathbf{b}, 0}(q) = \sum_{\tau \in \mathfrak{S}_k} (-q)^{\ell(w_0^{(k)} \tau)} t_{g \cdot \tau, f \cdot w_0^{(k)}}^{(\mathbf{b}, 0^k)}(q).$$

Proof. We write $w_0 = w_0^{(k)}$ in this proof. We identify \mathcal{V}_h in $\wedge^k \mathbb{V}$ with $M_{h \cdot w_0}^{(0^k)} H_0 \in \mathbb{V}^{\otimes k}$, for $h \in \mathbb{Z}_+^k$, so that the $\mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^k \mathbb{V}$ may be identified with a subspace of $\mathbb{T}^{\mathbf{b}} \widehat{\otimes} \mathbb{V}^{\otimes k}$. Then,

as in [Br1, Lemma 3.8], we have

$$T_f^{\mathbf{b},0} = T_{f \cdot w_0}^{(\mathbf{b},0^k)} H_0.$$

A straightforward variation of [Br1, Lemma 3.4] using (4.11) gives us

$$\begin{aligned} T_f^{\mathbf{b},0} &= T_{f \cdot w_0}^{(\mathbf{b},0^k)} H_0 = \sum_g t_{g,f \cdot w_0}^{(\mathbf{b},0^k)} M_g^{(\mathbf{b},0^k)} H_0 \\ &= \sum_{\tau \in \mathfrak{S}_k} \sum_{g \in \mathbb{Z}^{m+n} \times \mathbb{Z}_+^k} t_{g \cdot \tau, f \cdot w_0}^{(\mathbf{b},0^k)} M_{g \cdot \tau}^{(\mathbf{b},0^k)} H_0 \\ &= \sum_{\tau \in \mathfrak{S}_k} \sum_{g \in \mathbb{Z}^{m+n} \times \mathbb{Z}_+^k} t_{g \cdot \tau, f \cdot w_0}^{(\mathbf{b},0^k)} (-q)^{\ell(\tau^{-1}w_0)} M_g^{\mathbf{b},0} \\ &= \sum_{g \in \mathbb{Z}^{m+n} \times \mathbb{Z}_+^k} \left(\sum_{\tau \in \mathfrak{S}_k} t_{g \cdot \tau, f \cdot w_0}^{(\mathbf{b},0^k)} (-q)^{\ell(w_0\tau)} \right) M_g^{\mathbf{b},0}. \end{aligned}$$

The proposition now follows by comparing with (4.12). \square

Remark 4.11. Entirely analogous statements as Propositions 4.9 and 4.10 hold when comparing the canonical and dual canonical bases in $\mathbb{T}^{\mathbf{b}} \widehat{\otimes} \mathbb{W}^{\otimes k}$ with their counterparts in $\mathbb{T}^{\mathbf{b}} \widehat{\otimes} \wedge^k \mathbb{W}$. Such relations between Kazhdan-Lusztig polynomials and their parabolic versions were due to [Deo] (see [So1, Proposition 3.4]).

5. CANONICAL BASES FOR ADJACENT FOCK SPACES

In this section, we develop a new approach to compare the canonical as well as dual canonical bases in Fock spaces $\mathbb{T}^{\mathbf{b}}$ and $\mathbb{T}^{\mathbf{b}'}$, for adjacent $0^m 1^n$ -sequences \mathbf{b} and \mathbf{b}' .

5.1. Rank 2 cases. Consider the Fock space $\mathbb{T}^{(0,1)} = \mathbb{V} \otimes \mathbb{W}$ and its B -completion $\mathbb{V} \widehat{\otimes} \mathbb{W}$ with respect to the Bruhat ordering $\succeq_{(0,1)}$. It has the standard monomial basis $M_f := M_f^{(0,1)} = v_{f(1)} \otimes w_{f(2)}$, canonical basis $T_f := T_f^{(0,1)}$, and dual canonical basis $L_f := L_f^{(0,1)}$, for $f \in \mathbb{Z}^{1+1}$. For $f \in \mathbb{Z}^{1+1}$ with $f(1) = f(2)$ we introduce the following elements in \mathbb{Z}^{1+1} :

$$\begin{aligned} (5.1) \quad f^\downarrow(1) &= f^\downarrow(2) = f(1) - 1, \\ f^\uparrow(1) &= f^\uparrow(2) = f(1) + 1. \end{aligned}$$

Define $f^{\downarrow k} = (((f^\downarrow)^\downarrow) \dots^\downarrow)$, i.e. applying the operation \downarrow k times to f . Similarly, we introduce the notation $f^{\uparrow k}$.

Lemma 5.1. [Br1, Example 2.19] *We have the following formulas for the canonical and dual canonical bases in $\mathbb{V} \widehat{\otimes} \mathbb{W}$:*

$$\begin{aligned} L_f &= \begin{cases} M_f + \sum_{k=1}^{\infty} (-q)^{-k} M_{f \downarrow k}, & \text{if } f(1) = f(2), \\ M_f, & \text{if } f(1) \neq f(2); \end{cases} \\ T_f &= \begin{cases} M_f + q M_{f \downarrow}, & \text{if } f(1) = f(2), \\ M_f, & \text{if } f(1) \neq f(2). \end{cases} \end{aligned}$$

Therefore we have the inversion formula

$$(5.2) \quad M_f = \begin{cases} L_f + q^{-1} L_{f \downarrow}, & \text{if } f(1) = f(2), \\ L_f, & \text{if } f(1) \neq f(2). \end{cases}$$

Definition 5.2. Let \mathbb{L} be the $\mathbb{Q}(q)$ -subspace of $\mathbb{V} \widehat{\otimes} \mathbb{W}$ spanned by $\{L_f | f \in \mathbb{Z}^{1+1}\}$. Also, let \mathbb{U} be the $\mathbb{Q}(q)$ -subspace of $\mathbb{V} \widehat{\otimes} \mathbb{W}$ spanned by $\{T_f | f \in \mathbb{Z}^{1+1}\}$.

It can be checked directly that applying the Chevalley generators E_a and F_a to T_f produces a finite linear combination of T_g 's. This implies that \mathbb{U} is a $U_q(\mathfrak{gl}_{\infty})$ -module, and hence $(\mathbb{U}, \{T_f | f \in \mathbb{Z}^{1+1}\})$ is a weakly based module. Similarly, applying the Chevalley generators E_a and F_a to L_f produces a finite linear combination of M_g 's. This implies by the inversion formula (5.2) that $(\mathbb{L}, \{L_f | f \in \mathbb{Z}^{1+1}\})$ is also a weakly based $U_q(\mathfrak{gl}_{\infty})$ -module.

Next consider the $\mathbb{Q}(q)$ -space $\mathbb{T}^{(1,0)} = \mathbb{W} \otimes \mathbb{V}$ and its B -completion $\mathbb{W} \widehat{\otimes} \mathbb{V}$ with respect to the Bruhat ordering $\succeq_{(1,0)}$. It has the standard monomial basis $M'_f := M_f^{(1,0)} = w_{f(1)} \otimes v_{f(2)}$, canonical basis $T'_f := T_f^{(1,0)}$, and dual canonical basis $L'_f := L_f^{(1,0)}$, for $f \in \mathbb{Z}^{1+1}$. We have the following formulas in $\mathbb{W} \widehat{\otimes} \mathbb{V}$ similar to Lemma 5.1:

$$\begin{aligned} L'_f &= \begin{cases} M'_f + \sum_{k=1}^{\infty} (-q)^{-k} M'_{(f \uparrow k)}, & \text{if } f(1) = f(2), \\ M'_f, & \text{if } f(1) \neq f(2); \end{cases} \\ T'_f &= \begin{cases} M'_f + q M'_{f \uparrow}, & \text{if } f(1) = f(2), \\ M'_f, & \text{if } f(1) \neq f(2). \end{cases} \end{aligned}$$

Definition 5.3. Let \mathbb{L}' and \mathbb{U}' be the $\mathbb{Q}(q)$ -subspaces of $\mathbb{W} \widehat{\otimes} \mathbb{V}$ spanned by the sets $\{L'_f | f \in \mathbb{Z}^{1+1}\}$ and $\{T'_f | f \in \mathbb{Z}^{1+1}\}$, respectively.

It can be checked similarly as before that \mathbb{L}' and \mathbb{U}' are also weakly based $U_q(\mathfrak{gl}_{\infty})$ -modules. For $f \in \mathbb{Z}^{1+1}$, define $f \cdot \tau \in \mathbb{Z}^{1+1}$ by $(f \cdot \tau)(1) = f(2)$ and $(f \cdot \tau)(2) = f(1)$. Define a $\mathbb{Q}(q)$ -vector space isomorphism $\mathcal{R}_L : \mathbb{L} \rightarrow \mathbb{L}'$ by

$$\mathcal{R}_L(L_f) = \begin{cases} L'_{f \uparrow}, & \text{if } f(1) = f(2), \\ L'_{f \cdot \tau}, & \text{if } f(1) \neq f(2). \end{cases}$$

Similarly, define a $\mathbb{Q}(q)$ -vector space isomorphism $\mathcal{R}_U : \mathbb{U} \rightarrow \mathbb{U}'$ by

$$\mathcal{R}_U(T_f) = \begin{cases} T'_{f \downarrow}, & \text{if } f(1) = f(2), \\ T'_{f \cdot \tau}, & \text{if } f(1) \neq f(2). \end{cases}$$

Lemma 5.4. *The maps $\mathcal{R}_L : \mathbb{L} \rightarrow \mathbb{L}'$ and $\mathcal{R}_U : \mathbb{U} \rightarrow \mathbb{U}'$ are isomorphisms of weakly based $U_q(\mathfrak{gl}_\infty)$ -modules.*

Proof. This can be checked by considering the cases one by one. The calculations are fairly easy, and as an illustration, we shall do this only for two non-trivial cases for \mathcal{R}_L below. Let us identify $f \in \mathbb{Z}^{1+1}$ with the tuple $(f(1), f(2))$.

In the first case, for any $a \in \mathbb{Z}$ we have

$$\mathcal{R}_L(E_a L_{a,a}) = \mathcal{R}_L(v_a \otimes w_{a+1}) = w_{a+1} \otimes v_a = E_a L'_{a+1,a+1} = E_a \mathcal{R}_L(L_{a,a}).$$

In the second case, for any $a \in \mathbb{Z}$,

$$\begin{aligned} \mathcal{R}_L(F_a L_{a,a+1}) &= \mathcal{R}_L(\Delta(F_a)(v_a \otimes w_{a+1})) = \mathcal{R}_L(v_{a+1} \otimes w_{a+1} + qv_a \otimes w_a) \\ &= \mathcal{R}_L(L_{a+1,a+1} + q^{-1}L_{a,a} + qL_{a,a} + L_{a-1,a-1}) \\ &= L'_{a+2,a+2} + q^{-1}L'_{a+1,a+1} + qL'_{a+1,a+1} + L'_{a,a}. \end{aligned}$$

On the other hand, we compute

$$\begin{aligned} F_a \mathcal{R}_L(L_{a,a+1}) &= F_a \mathcal{R}_L(v_a \otimes w_{a+1}) = \Delta(F_a)(w_{a+1} \otimes v_a) = w_a \otimes v_a + qw_{a+1} \otimes v_{a+1} \\ &= L'_{a,a} + q^{-1}L'_{a+1,a+1} + qL'_{a+1,a+1} + L'_{a+2,a+2}. \end{aligned}$$

So $\mathcal{R}_L(F_a L_{a,a+1}) = F_a \mathcal{R}_L(L_{a,a+1})$. □

Remark 5.5. The definitions of the maps \mathcal{R}_L and \mathcal{R}_U are motivated and compatible in a suitable sense with the map from $\mathbb{V} \otimes \mathbb{W}$ to $\mathbb{W} \hat{\otimes} \mathbb{V}$ given by the \mathcal{R} -matrix. Explicitly, the map on the standard monomial basis is given as follows:

$$M_f \mapsto \begin{cases} q^{-1}M'_f - (q - q^{-1}) \sum_{k \geq 1} (-q)^{-k} M'_{f \uparrow k}, & \text{if } f(1) = f(2), \\ M'_f, & \text{if } f(1) \neq f(2). \end{cases}$$

However, such an \mathcal{R} -map does not extend to the B -completion $\mathbb{V} \hat{\otimes} \mathbb{W}$. This is why we have to work with \mathcal{R}_L and \mathcal{R}_U instead, and then with suitable completions using \mathbb{L} and \mathbb{U} in the next subsections.

5.2. Canonical basis revisited. In this subsection, we shall give another description of canonical and dual canonical bases in the Fock spaces.

Fix a $0^m 1^n$ -sequence of the form $\mathbf{b} = (\mathbf{b}^1, 0, 1, \mathbf{b}^2)$, where \mathbf{b}^1 and \mathbf{b}^2 are $0^{m_1} 1^{n_1}$ - and $0^{m_2} 1^{n_2}$ -sequences satisfying $m = m_1 + m_2 + 1$ and $n = n_1 + n_2 + 1$. Set

$$(5.3) \quad \kappa = m_1 + n_1 + 1.$$

Recalling the $U_q(\mathfrak{gl}_\infty)$ -module \mathbb{L} from Definition 5.2, we form a $\mathbb{Q}(q)$ -vector space

$$(5.4) \quad \mathbb{T}_{\mathbb{L}} := \mathbb{T}^{\mathbf{b}^1} \otimes \mathbb{L} \otimes \mathbb{T}^{\mathbf{b}^2},$$

which admits the following basis

$$N_f := v_{f(1)}^{b_1} \otimes \cdots \otimes v_{f(\kappa-1)}^{b_{\kappa-1}} \otimes L_{f_{\kappa}, \kappa+1} \otimes v_{f(\kappa+2)}^{b_{\kappa+2}} \otimes \cdots \otimes v_{f(m+n)}^{b_{m+n}}, \quad \text{for } f \in \mathbb{Z}^{m+n}.$$

Extending the definition (5.1) for $m = n = 1$, to $f \in \mathbb{Z}^{m+n}$ with $f(\kappa) = f(\kappa + 1)$, we define $f^\downarrow, f^\uparrow \in \mathbb{Z}^{m+n}$ such that

$$(5.5) \quad \begin{aligned} f^\uparrow(i) &= f^\downarrow(i) = f(i), \quad \text{for } i \neq \kappa, \kappa + 1, \\ f^\uparrow(\kappa) &= f^\uparrow(\kappa + 1) = f(\kappa) + 1, \quad \text{and } f^\downarrow(\kappa) = f^\downarrow(\kappa + 1) = f(\kappa) - 1. \end{aligned}$$

It follows by definition and (5.2) that

$$(5.6) \quad M_f^{\mathbf{b}} = \begin{cases} N_f + q^{-1}N_{f^\downarrow}, & \text{if } f(\kappa) = f(\kappa + 1), \\ N_f, & \text{if } f(\kappa) \neq f(\kappa + 1). \end{cases}$$

As in Lemma 5.1, we can write $N_f \in \tilde{\mathbb{T}}^{\mathbf{b}}$ as

$$(5.7) \quad N_f = \begin{cases} M_f^{\mathbf{b}} + \sum_{k=1}^{\infty} (-q)^{-k} M_{f^\downarrow k}^{\mathbf{b}}, & \text{if } f(\kappa) = f(\kappa + 1), \\ M_f^{\mathbf{b}}, & \text{if } f(\kappa) \neq f(\kappa + 1). \end{cases}$$

Since \mathbb{L} is a $U_q(\mathfrak{gl}_\infty)$ -module, we can write $\overline{N}_f = \sum_g a_{gf} N_g$ as an element in the A -completion $\hat{\mathbb{T}}_{\mathbb{L}}$, which is defined similarly as in Definition 3.2. Recalling that $\overline{M}_g^{\mathbf{b}}$ is of the form $\overline{M}_g^{\mathbf{b}} = \sum_{h \leq_{\mathbf{b}} g} r_{hg} M_h^{\mathbf{b}}$, for $r_{hg} \in \mathbb{Z}[q, q^{-1}]$, we conclude from (5.6) and (5.7) that

$$(5.8) \quad \overline{N}_f = N_f + \sum_{g \prec_{\mathbf{b}} f} a_{gf} N_g, \quad \text{for } a_{gf} \in \mathbb{Z}[q, q^{-1}].$$

We form the B -completion $\hat{\mathbb{T}}_{\mathbb{L}}$ of $\mathbb{T}_{\mathbb{L}}$, which is the $\mathbb{Q}(q)$ -vector space spanned by elements of the form $N_f + \sum_{g \prec_{\mathbf{b}} f} d_{gf} N_g$, following Definition 3.2. Note that $\overline{N}_f \in \hat{\mathbb{T}}_{\mathbb{L}}$ by (5.8).

The following lemma follows directly from (5.8) and Lemma 3.8.

Lemma 5.6. *There exists a unique bar-invariant topological basis $\{L_f | f \in \mathbb{Z}^{m+n}\}$ in $\hat{\mathbb{T}}_{\mathbb{L}}$ such that*

$$(5.9) \quad L_f = \sum_{g \leq_{\mathbf{b}} f} \check{\ell}_{gf}(q) N_g,$$

where $\check{\ell}_{ff} = 1$ and $\check{\ell}_{gf}(q) \in q^{-1}\mathbb{Z}[q^{-1}]$, for $g \prec_{\mathbf{b}} f$.

We call $\{L_f | f \in \mathbb{Z}^{m+n}\}$ the *dual canonical basis* of $\hat{\mathbb{T}}_{\mathbb{L}}$.

Recalling now \mathbb{U} from Definition 5.2, we form the $\mathbb{Q}(q)$ -vector space

$$(5.10) \quad \mathbb{T}_{\mathbb{U}} := \mathbb{T}^{\mathbf{b}^1} \otimes \mathbb{U} \otimes \mathbb{T}^{\mathbf{b}^2},$$

which has the following basis

$$U_f := \mathbf{v}_{f(1)}^{b_1} \otimes \cdots \otimes \mathbf{v}_{f(\kappa-1)}^{b_{\kappa-1}} \otimes T_{f_{\kappa, \kappa+1}} \otimes \mathbf{v}_{f(\kappa+2)}^{b_{\kappa+2}} \otimes \cdots \otimes \mathbf{v}_{f(m+n)}^{b_{m+n}}, \quad \text{for } f \in \mathbb{Z}^{m+n}.$$

It follows by definition and Lemma 5.1 that

$$(5.11) \quad U_f = \begin{cases} M_f^{\mathbf{b}} + qM_{f^\downarrow}^{\mathbf{b}}, & \text{if } f(\kappa) = f(\kappa + 1), \\ M_f^{\mathbf{b}}, & \text{if } f(\kappa) \neq f(\kappa + 1). \end{cases}$$

By a similar argument as for (5.8) we have

$$(5.12) \quad \overline{U}_f = \sum_{g \preceq_{\mathbf{b}} f} d_{gf} U_g, \quad d_{gf} \in \mathbb{Z}[q, q^{-1}].$$

Let $\widehat{\mathbb{T}}_{\mathbb{U}}$ denote the B -completion of $\mathbb{T}_{\mathbb{U}}$, following Definition 3.2. Then $\overline{U}_f \in \widehat{\mathbb{T}}_{\mathbb{U}}$ by (5.12).

The following lemma is immediate from (5.12) and Lemma 3.8.

Lemma 5.7. *There exists a unique bar-invariant topological basis $\{T_f | f \in \mathbb{Z}^{m+n}\}$ in $\widehat{\mathbb{T}}_{\mathbb{U}}$ such that*

$$(5.13) \quad T_f = \sum_{g \preceq_{\mathbf{b}} f} \check{t}_{gf}(q) U_g,$$

where $\check{t}_{ff} = 1$ and $\check{t}_{gf}(q) \in q\mathbb{Z}[q]$, for $g \prec_{\mathbf{b}} f$.

We call $\{T_f | f \in \mathbb{Z}^{m+n}\}$ the *canonical basis* of $\widehat{\mathbb{T}}_{\mathbb{U}}$.

Recall that $\{L_f^{\mathbf{b}} | f \in \mathbb{Z}^{m+n}\}$ denotes the dual canonical basis and $\{T_f^{\mathbf{b}} | f \in \mathbb{Z}^{m+n}\}$ denotes the canonical basis in $\widehat{\mathbb{T}}^{\mathbf{b}}$. Note that $\widehat{\mathbb{T}}_{\mathbb{U}} \subseteq \widehat{\mathbb{T}}^{\mathbf{b}}$ by definition and (5.7). Similarly, we have $\widehat{\mathbb{T}}_{\mathbb{U}} \subseteq \widehat{\mathbb{T}}^{\mathbf{b}}$. Hence we may naturally regard $L_f \in \widehat{\mathbb{T}}_{\mathbb{L}}$ as an element in $\widehat{\mathbb{T}}^{\mathbf{b}}$ and $T_f \in \widehat{\mathbb{T}}_{\mathbb{U}}$ as an element in $\widehat{\mathbb{T}}^{\mathbf{b}}$.

Proposition 5.8. *We have the following identification of canonical and dual canonical bases: $L_f = L_f^{\mathbf{b}} \in \widehat{\mathbb{T}}_{\mathbb{L}} \subseteq \widehat{\mathbb{T}}^{\mathbf{b}}$ and $T_f = T_f^{\mathbf{b}} \in \widehat{\mathbb{T}}_{\mathbb{U}} \subseteq \widehat{\mathbb{T}}^{\mathbf{b}}$, for $f \in \mathbb{Z}^{m+n}$.*

Proof. The proofs of the two identities are analogous, and we will prove the first one.

Recall that we have $L_f^{\mathbf{b}} = M_f^{\mathbf{b}} + \sum_{g \prec_{\mathbf{b}} f} \ell_{gf}^{\mathbf{b}}(q) M_g^{\mathbf{b}}$, where $\ell_{gf}^{\mathbf{b}}(q) \in q^{-1}\mathbb{Z}[q^{-1}]$. Using (5.7) and (5.9) we obtain an expression for the bar-invariant element L_f that equals M_f plus a $q^{-1}\mathbb{Z}[q^{-1}]$ -linear combination of M_g with $g \prec_{\mathbf{b}} f$. By the uniqueness of the dual canonical basis in $\widehat{\mathbb{T}}^{\mathbf{b}}$ (see Lemma 3.8), this must be equal to $L_f^{\mathbf{b}}$. Hence, $L_f = L_f^{\mathbf{b}}$. \square

We call $\{N_f | f \in \mathbb{Z}^{m+n}\}$ and $\{U_f | f \in \mathbb{Z}^{m+n}\}$ the *parabolic monomial bases* for $\widehat{\mathbb{T}}^{\mathbf{b}}$.

Remark 5.9. It is natural to conjecture that the polynomials $\check{\ell}_{gf}(q)$ defined in Lemma 5.6 and the polynomials $\check{t}_{gf}(q)$ in Lemma 5.7 satisfy the positivity property: $\check{t}_{gf}(q) \in \mathbb{N}[q]$ and $\check{\ell}_{gf}(-q^{-1}) \in \mathbb{N}[q]$. In light of (5.7) and (5.11), the positivity of $\check{t}_{gf}(q)$ and $\check{\ell}_{gf}(-q^{-1})$ implies Conjecture 3.13.

Two $0^m 1^n$ -sequences are said to be *adjacent* to each other if they are identical except for a switch of a neighboring pair $\{0, 1\}$, that is, the $0^m 1^n$ -sequences $\mathbf{b}' := (\mathbf{b}^1, 1, 0, \mathbf{b}^2)$ and $\mathbf{b} = (\mathbf{b}^1, 0, 1, \mathbf{b}^2)$ are adjacent. The constructions below for \mathbf{b}' are analogous to the above constructions for \mathbf{b} , so we will merely set up the necessary notations for later use. Recall the spaces \mathbb{L}' and \mathbb{U}' from Definition 5.3. We form the $\mathbb{Q}(q)$ -vector space

$$(5.14) \quad \mathbb{T}'_{\mathbb{L}} := \mathbb{T}^{\mathbf{b}^1} \otimes \mathbb{L}' \otimes \mathbb{T}^{\mathbf{b}^2},$$

which admits a *parabolic monomial basis*

$$N'_f := \mathbf{v}_{f(1)}^{b_1} \otimes \cdots \otimes \mathbf{v}_{f(\kappa-1)}^{b_{\kappa-1}} \otimes L'_{f_{\kappa, \kappa+1}} \otimes \mathbf{v}_{f(\kappa+2)}^{b_{\kappa+2}} \otimes \cdots \otimes \mathbf{v}_{f(m+n)}^{b_{m+n}}, \quad \text{for } f \in \mathbb{Z}^{m+n}.$$

Here we recall κ from (5.3). We proceed as before to define the B -completion $\widehat{\mathbb{T}}'_\mathbb{L}$ of $\mathbb{T}'_\mathbb{L}$, and then obtain *dual canonical basis elements* in $\widehat{\mathbb{T}}'_\mathbb{L}$ denoted accordingly by $\{L'_f\}$. Note that $\widehat{\mathbb{T}}'_\mathbb{L} \subseteq \widehat{\mathbb{T}}^{\mathbf{b}'}$.

In addition, we form the $\mathbb{Q}(q)$ -vector space

$$(5.15) \quad \mathbb{T}'_\mathbb{U} := \mathbb{T}^{\mathbf{b}^1} \otimes \mathbb{U}' \otimes \mathbb{T}^{\mathbf{b}^2}.$$

which has a *parabolic monomial basis*

$$U'_f := \mathbf{v}_{f(1)}^{b_1} \otimes \cdots \otimes \mathbf{v}_{f(\kappa-1)}^{b_{\kappa-1}} \otimes T'_{f\kappa, \kappa+1} \otimes \mathbf{v}_{f(\kappa+2)}^{b_{\kappa+2}} \otimes \cdots \otimes \mathbf{v}_{f(m+n)}^{b_{m+n}}, \quad \text{for } f \in \mathbb{Z}^{m+n}.$$

We proceed as before to define the B -completion $\widehat{\mathbb{T}}'_\mathbb{U}$ of $\mathbb{T}'_\mathbb{U}$, and then obtain the *canonical basis* in $\widehat{\mathbb{T}}'_\mathbb{U}$ denoted accordingly by $\{T'_f\}$. Note that $\widehat{\mathbb{T}}'_\mathbb{U} \subseteq \widehat{\mathbb{T}}^{\mathbf{b}'}$.

Hence we may naturally regard $L'_f \in \widehat{\mathbb{T}}'_\mathbb{L}$ as an element in $\widehat{\mathbb{T}}^{\mathbf{b}'}$ and $T'_f \in \widehat{\mathbb{T}}'_\mathbb{U}$ as an element in $\widehat{\mathbb{T}}^{\mathbf{b}'}$. The following analogue of Propositions 5.8 can be proved in exactly the same way.

Proposition 5.10. *We have the following identification of canonical and dual canonical bases: $L'_f = L^{\mathbf{b}'}_f \in \widehat{\mathbb{T}}'_\mathbb{L} \subseteq \widehat{\mathbb{T}}^{\mathbf{b}'}$ and $T'_f = T^{\mathbf{b}'}_f \in \widehat{\mathbb{T}}'_\mathbb{U} \subseteq \widehat{\mathbb{T}}^{\mathbf{b}'}$, for $f \in \mathbb{Z}^{m+n}$.*

5.3. Adjacent canonical bases. We continue to use the notations of the adjacent sequences \mathbf{b} and \mathbf{b}' as well as κ from §5.2.

For $f \in \mathbb{Z}^{m+n}$, define $f \cdot \tau \in \mathbb{Z}^{m+n}$ by letting

$$\begin{aligned} (f \cdot \tau)(i) &= f(i), \quad \text{for } i \neq \kappa, \kappa + 1, \\ (f \cdot \tau)(\kappa) &= f(\kappa + 1), \quad \text{and } (f \cdot \tau)(\kappa + 1) = f(\kappa). \end{aligned}$$

Recall f^\uparrow, f^\downarrow from (5.5). The following notations will be convenient in the sequel: for $f \in \mathbb{Z}^{m+n}$ set

$$(5.16) \quad f^\mathbb{L} = \begin{cases} f \cdot \tau, & \text{if } f(\kappa) \neq f(\kappa + 1), \\ f^\uparrow, & \text{if } f(\kappa) = f(\kappa + 1), \end{cases}$$

$$(5.17) \quad f^\mathbb{U} = \begin{cases} f \cdot \tau, & \text{if } f(\kappa) \neq f(\kappa + 1), \\ f^\downarrow, & \text{if } f(\kappa) = f(\kappa + 1). \end{cases}$$

Recall from Lemma 5.4 the $U_q(\mathfrak{gl}_\infty)$ -module isomorphism \mathcal{R}_L of the weakly based modules \mathbb{L} and \mathbb{L}' . Recall $\mathbb{T}_\mathbb{L} = \mathbb{T}^{\mathbf{b}^1} \otimes \mathbb{L} \otimes \mathbb{T}^{\mathbf{b}^2}$ and $\mathbb{T}'_\mathbb{L} = \mathbb{T}^{\mathbf{b}^1} \otimes \mathbb{L}' \otimes \mathbb{T}^{\mathbf{b}^2}$. Then

$$\begin{aligned} \mathcal{R} &\stackrel{\text{def}}{=} 1_{\mathbf{b}^1} \otimes \mathcal{R}_L \otimes 1_{\mathbf{b}^2} : \mathbb{T}_\mathbb{L} \longrightarrow \mathbb{T}'_\mathbb{L} \\ \mathcal{R}(N_f) &= N'_{f^\mathbb{L}}, \quad \forall f, \end{aligned}$$

is an isomorphism of $U_q(\mathfrak{gl}_\infty)$ -modules.

Define the truncated subspaces $[\mathbb{T}_\mathbb{L}]_{\leq |k|}$ and $[\mathbb{T}'_\mathbb{L}]_{\leq |k|}$ for $k \in \mathbb{N}$ as in §3.2, and then form the topological A -completions $\widetilde{\mathbb{T}}_\mathbb{L}$ and $\widetilde{\mathbb{T}}'_\mathbb{L}$ as in Definition 3.2 with corresponding projection maps $\pi_{\mathbb{L},k} : \mathbb{T}_\mathbb{L} \rightarrow [\mathbb{T}_\mathbb{L}]_{\leq |k|}$ and $\pi'_{\mathbb{L},k} : \mathbb{T}'_\mathbb{L} \rightarrow [\mathbb{T}'_\mathbb{L}]_{\leq |k|}$ as in (3.6). We have $\mathcal{R}([\mathbb{T}_\mathbb{L}]_{\leq |k|}) \subseteq [\mathbb{T}'_\mathbb{L}]_{\leq |k+1|}$, and hence \mathcal{R} extends to a linear isomorphism $\mathcal{R} : \widetilde{\mathbb{T}}_\mathbb{L} \rightarrow \widetilde{\mathbb{T}}'_\mathbb{L}$, which is actually a homeomorphism of topological vector spaces.

- Definition 5.11.** (1) The *partial ordering* $\preceq_{\mathbf{b}, \mathbf{b}'}$ on \mathbb{Z}^{m+n} is defined as follows:
 $g \preceq_{\mathbf{b}, \mathbf{b}'} f$ if and only if $g \preceq_{\mathbf{b}} f$ and $g^{\mathbb{L}} \preceq_{\mathbf{b}'} f^{\mathbb{L}}$, for $f, g \in \mathbb{Z}^{m+n}$.
(2) The *partial ordering* $\preceq_{\mathbf{b}, \mathbf{b}'}^*$ on \mathbb{Z}^{m+n} is defined as follows: $g \preceq_{\mathbf{b}, \mathbf{b}'}^* f$ if and only if $g \preceq_{\mathbf{b}'} f$ and $g^{\mathbb{U}} \preceq_{\mathbf{b}} f^{\mathbb{U}}$, for $f, g \in \mathbb{Z}^{m+n}$.
(3) The *C-completion* of $\mathbb{T}_{\mathbb{L}}$, denoted by $\ddot{\mathbb{T}}_{\mathbb{L}}$, is the $\mathbb{Q}(q)$ -subspace of $\tilde{\mathbb{T}}_{\mathbb{L}}$ spanned by vectors of the form $N_f + \sum_{g \prec_{\mathbf{b}, \mathbf{b}'} f} r_g N_g$, for $r_g \in \mathbb{Q}(q)$.
(4) The *C-completion* of $\mathbb{T}'_{\mathbb{L}}$, denoted by $\ddot{\mathbb{T}}'_{\mathbb{L}}$, is the $\mathbb{Q}(q)$ -subspace of $\tilde{\mathbb{T}}'_{\mathbb{L}}$ spanned by vectors of the form $N'_f + \sum_{g \prec_{\mathbf{b}, \mathbf{b}'}^* f} r_g N_g$, for $r_g \in \mathbb{Q}(q)$.

In other words, $\ddot{\mathbb{T}}_{\mathbb{L}}$ is simply the *B-completion* of $\mathbb{T}_{\mathbb{L}}$ with respect to the refined partial ordering $\preceq_{\mathbf{b}, \mathbf{b}'}$, while $\ddot{\mathbb{T}}'_{\mathbb{L}}$ is the *B-completion* of $\mathbb{T}'_{\mathbb{L}}$ with respect to $\preceq_{\mathbf{b}, \mathbf{b}'}^*$. By definition, the *A*-, *B*- and *C*-completions of $\mathbb{T}_{\mathbb{L}}$ and $\mathbb{T}'_{\mathbb{L}}$ in (5.4) and (5.14) are related as follows:

$$\ddot{\mathbb{T}}_{\mathbb{L}} \subseteq \widehat{\mathbb{T}}_{\mathbb{L}} \subseteq \tilde{\mathbb{T}}_{\mathbb{L}}, \quad \ddot{\mathbb{T}}'_{\mathbb{L}} \subseteq \widehat{\mathbb{T}}'_{\mathbb{L}} \subseteq \tilde{\mathbb{T}}'_{\mathbb{L}}.$$

Since $(f^{\mathbb{L}})^{\mathbb{U}} = (f^{\mathbb{U}})^{\mathbb{L}} = f$, we have

$$(5.18) \quad g \prec_{\mathbf{b}, \mathbf{b}'} f \text{ if and only if } g^{\mathbb{L}} \prec_{\mathbf{b}, \mathbf{b}'}^* f^{\mathbb{L}}.$$

Theorem 5.12. *Let \mathbf{b} and \mathbf{b}' be adjacent $0^m 1^n$ -sequences. Then*

- (1) *the restriction of $\mathcal{R} : \tilde{\mathbb{T}}_{\mathbb{L}} \xrightarrow{\sim} \tilde{\mathbb{T}}'_{\mathbb{L}}$ gives a $\mathbb{Q}(q)$ -linear isomorphism $\mathcal{R} : \ddot{\mathbb{T}}_{\mathbb{L}} \rightarrow \ddot{\mathbb{T}}'_{\mathbb{L}}$;*
- (2) *$\ddot{\mathbb{T}}_{\mathbb{L}}$ and $\ddot{\mathbb{T}}'_{\mathbb{L}}$ are bar-invariant subspaces of $\widehat{\mathbb{T}}_{\mathbb{L}}$ and $\widehat{\mathbb{T}}'_{\mathbb{L}}$, respectively. Moreover, the dual canonical bases of $\widehat{\mathbb{T}}_{\mathbb{L}}$ and $\widehat{\mathbb{T}}'_{\mathbb{L}}$ lie in $\ddot{\mathbb{T}}_{\mathbb{L}}$ and $\ddot{\mathbb{T}}'_{\mathbb{L}}$, respectively;*
- (3) *$\mathcal{R}(N_f) = N'_{f^{\mathbb{L}}}$, $\mathcal{R}(L_f) = L'_{f^{\mathbb{L}}}$, and $\mathcal{R}(L_f^{\mathbf{b}}) = L'_{f^{\mathbb{L}}}^{\mathbf{b}'}$, for all $f \in \mathbb{Z}^{m+n}$.*

Proof. By definition, we have

$$(5.19) \quad \mathcal{R}(N_f) = N'_{f^{\mathbb{L}}}, \quad \forall f.$$

Part (1) follows from this, (5.18), and the definition of *C*-completions $\ddot{\mathbb{T}}_{\mathbb{L}}$, $\ddot{\mathbb{T}}'_{\mathbb{L}}$.

We now first work with the *A*-completions and with $\mathcal{R} : \tilde{\mathbb{T}}_{\mathbb{L}} \rightarrow \tilde{\mathbb{T}}'_{\mathbb{L}}$. It follows from (5.19) and the definition of the bar map on tensor modules (cf. (3.8) and (3.10)) that

$$(5.20) \quad \mathcal{R}(\overline{N_f}) = \overline{\mathcal{R}(N_f)} = \overline{N'_{f^{\mathbb{L}}}}.$$

Hence from (5.8), (5.19) and (5.20) we obtain that

$$(5.21) \quad \overline{N'_{f^{\mathbb{L}}}} = N'_{f^{\mathbb{L}}} + \sum_{g \prec_{\mathbf{b}} f} a_{gf} N'_{g^{\mathbb{L}}}.$$

On the other hand, $\overline{N'_{f^{\mathbb{L}}}} \in \widehat{\mathbb{T}}'_{\mathbb{L}} \subseteq \tilde{\mathbb{T}}'_{\mathbb{L}}$ can be written in the form

$$(5.22) \quad \overline{N'_{f^{\mathbb{L}}}} = N'_{f^{\mathbb{L}}} + \sum_{g^{\mathbb{L}} \prec_{\mathbf{b}'} f^{\mathbb{L}}} a'_{gf} N'_{g^{\mathbb{L}}}.$$

The comparison between (5.21) and (5.22) implies that $a'_{gf} = a_{gf}$ if $g \prec_{\mathbf{b}, \mathbf{b}'} f$, and

$$(5.23) \quad \overline{N'_{f^{\mathbb{L}}}} = N'_{f^{\mathbb{L}}} + \sum_{g \prec_{\mathbf{b}, \mathbf{b}'} f} a_{gf} N'_{g^{\mathbb{L}}} = N'_{f^{\mathbb{L}}} + \sum_{g^{\mathbb{L}} \prec_{\mathbf{b}, \mathbf{b}'}^* f^{\mathbb{L}}} a_{gf} N'_{g^{\mathbb{L}}}.$$

Using the inverse \mathcal{R}^{-1} instead and arguing similarly as above, we then obtain the following counterpart of (5.23):

$$(5.24) \quad \overline{N_f} = N_f + \sum_{g \prec_{\mathbf{b}, \mathbf{b}'} f} a_{gf} N_g.$$

Hence, $\overline{N_f}$ actually lies in the C -completion $\ddot{\mathbb{T}}_{\mathbb{L}}$ (and then in B -completion $\widehat{\mathbb{T}}_{\mathbb{L}}$) and also $\overline{N'_{f\mathbb{L}}} \in \ddot{\mathbb{T}}'_{\mathbb{L}} \subset \widehat{\mathbb{T}}'_{\mathbb{L}}$. The first half of (2) now follows from (5.23) and (5.24).

Now we can work within the B -completions. By Lemma 5.6, (5.24), and the uniqueness part of Lemma 3.8 applied to the partial ordering $\preceq_{\mathbf{b}, \mathbf{b}'}$, the dual canonical basis element L_f in $\widehat{\mathbb{T}}_{\mathbb{L}}$ satisfies the refined partial ordering as follows:

$$(5.25) \quad L_f = N_f + \sum_{g \prec_{\mathbf{b}, \mathbf{b}'} f} \check{\ell}_{gf}(q) N_g,$$

where $\check{\ell}_{gf}(q) \in q^{-1}\mathbb{Z}[q^{-1}]$, for $g \prec_{\mathbf{b}, \mathbf{b}'} f$. This strengthens (5.9). Hence $L_f \in \ddot{\mathbb{T}}_{\mathbb{L}}$. Similarly, by Lemma 5.7 and (5.23) we have

$$(5.26) \quad L'_f = N'_f + \sum_{g \prec_{\mathbf{b}, \mathbf{b}'}^* f} \check{\ell}'_{gf}(q) N'_g,$$

where $\check{\ell}'_{gf}(q) \in q^{-1}\mathbb{Z}[q^{-1}]$, for $g \prec_{\mathbf{b}, \mathbf{b}'}^* f$, and hence $L'_f \in \ddot{\mathbb{T}}'_{\mathbb{L}}$. This proves the second part of (2).

Thanks to (5.19) and (5.25), $\mathcal{R}(L_f)$ satisfies the same characterization as the dual canonical basis element $L'_{f\mathbb{L}}$ (similar to Lemma 5.6). Hence $\mathcal{R}(L_f) = L'_{f\mathbb{L}}$ by the uniqueness of dual canonical basis.

Now $\mathcal{R}(L_f^{\mathbf{b}}) = L_{f\mathbb{L}}^{\mathbf{b}'}$ follows from the identifications in Propositions 5.8 and 5.10. \square

Corollary 5.13. *In the notations of (5.25) and (5.26), we have $\check{\ell}_{gf}(q) = \check{\ell}'_{g\mathbb{L}f\mathbb{L}}(q)$.*

Similarly, we recall $\mathcal{R}_U : \mathbb{U} \rightarrow \mathbb{U}'$ from Lemma 5.4, $\mathbb{T}_{\mathbb{U}} = \mathbb{T}^{\mathbf{b}_1} \otimes \mathbb{U} \otimes \mathbb{T}^{\mathbf{b}_2}$ from (5.10), and $\mathbb{T}'_{\mathbb{U}} = \mathbb{T}^{\mathbf{b}_1} \otimes \mathbb{U}' \otimes \mathbb{T}^{\mathbf{b}_2}$ from (5.15). Then we have an isomorphism of $U_q(\mathfrak{gl}_{\infty})$ -modules

$$\begin{aligned} \mathcal{R}^u &\stackrel{\text{def}}{=} 1_{\mathbf{b}_1} \otimes \mathcal{R}_U \otimes 1_{\mathbf{b}_2} : \mathbb{T}_{\mathbb{U}} \longrightarrow \mathbb{T}'_{\mathbb{U}} \\ \mathcal{R}^u(U_f) &= U'_{f\mathbb{U}}, \quad \forall f, \end{aligned}$$

The isomorphism \mathcal{R}^u extends to an isomorphism on the A -completions $\mathcal{R}^u : \widetilde{\mathbb{T}}_{\mathbb{U}} \rightarrow \widetilde{\mathbb{T}}'_{\mathbb{U}}$ as before. The C -completions $\ddot{\mathbb{T}}_{\mathbb{U}}$ and $\ddot{\mathbb{T}}'_{\mathbb{U}}$ of $\mathbb{T}_{\mathbb{U}}$ and $\mathbb{T}'_{\mathbb{U}}$ are defined as the B -completions of $\mathbb{T}_{\mathbb{U}}$ and $\mathbb{T}'_{\mathbb{U}}$ respectively with respect to some suitably refined partial orderings (given by the conditions in the sums (5.27) and (5.28) below).

Theorem 5.14. *Let \mathbf{b} and \mathbf{b}' be adjacent $0^m 1^n$ -sequences. Then*

- (1) *the restriction of $\mathcal{R}^u : \widetilde{\mathbb{T}}_{\mathbb{U}} \xrightarrow{\sim} \widetilde{\mathbb{T}}'_{\mathbb{U}}$ gives an isomorphism $\mathcal{R}^u : \ddot{\mathbb{T}}_{\mathbb{U}} \rightarrow \ddot{\mathbb{T}}'_{\mathbb{U}}$;*
- (2) *the canonical bases of $\widehat{\mathbb{T}}_{\mathbb{U}}$ and $\widehat{\mathbb{T}}'_{\mathbb{U}}$ lie in $\ddot{\mathbb{T}}_{\mathbb{U}}$ and $\ddot{\mathbb{T}}'_{\mathbb{U}}$, respectively;*
- (3) *$\mathcal{R}^u(U_f) = U'_{f\mathbb{U}}$, $\mathcal{R}^u(T_f) = T'_{f\mathbb{U}}$, and $\mathcal{R}^u(T_f^{\mathbf{b}}) = T_{f\mathbb{U}}^{\mathbf{b}'}$, for all $f \in \mathbb{Z}^{m+n}$.*

Theorem 5.14 is the canonical basis analogue of Theorem 5.12, where (3) uses the identification provided by Propositions 5.8 and 5.10. While we will skip the entirely analogous proof, we note that we obtain the following analogues of (5.25) and (5.26) in the process of proof:

$$(5.27) \quad T_f = U_f + \sum_{g \prec_{\mathbf{b}'f} g^{\mathbf{U}} \prec_{\mathbf{b}'f} g^{\mathbf{U}}} \check{t}_{gf}(q) U_g, \quad \text{where } \check{t}_{gf}(q) \in q\mathbb{Z}[q].$$

$$(5.28) \quad T'_f = U'_f + \sum_{g \prec_{\mathbf{b}'f} g^{\mathbf{L}} \prec_{\mathbf{b}'f} g^{\mathbf{L}}} \check{t}'_{gf}(q) U'_g, \quad \text{where } \check{t}'_{gf}(q) \in q\mathbb{Z}[q].$$

We also have the following corollary to Theorem 5.14.

Corollary 5.15. *In the notations of (5.27) and (5.28), we have $\check{t}_{gf}(q) = \check{t}'_{g^{\mathbf{U}}f^{\mathbf{U}}}(q)$.*

Part 2. Representation Theory

6. BGG CATEGORY FOR BASIC LIE SUPERALGEBRAS

In this section, we establish some basic properties of the BGG category \mathcal{O} of $\mathfrak{gl}(m|n)$ -modules. We show that the category \mathcal{O} is independent of the choice of non-conjugate Borel subalgebras. We then make systematic comparisons of the Verma, simple and tilting modules with respect to different Borel subalgebras. Finally, we introduce certain parabolic Verma modules associated to a pair of adjacent Borel subalgebras. All the results in this section remain valid for arbitrary basic Lie superalgebras.

6.1. Preliminaries. Let $\mathbb{C}^{m|n}$ be the complex superspace of dimension $(m|n)$. The general linear Lie superalgebra $\mathfrak{gl}(m|n)$ is the Lie superalgebra of linear transformations from $\mathbb{C}^{m|n}$ to itself. Thus, with respect to a given ordered basis of $\mathbb{C}^{m|n}$, $\mathfrak{gl}(m|n)$ may be realized in terms of $(m+n) \times (m+n)$ matrices over \mathbb{C} . Let $\{e_1, \dots, e_m\}$ and $\{e_{m+1}, \dots, e_{m+n}\}$ be the standard bases for the even subspace $\mathbb{C}^{m|0}$ and the odd subspace $\mathbb{C}^{0|n}$, respectively, so that their union is a homogeneous basis for $\mathbb{C}^{m|n}$. Then with respect to this ordered basis we let e_{ij} , $1 \leq i, j \leq m+n$, denote the (i, j) th elementary matrix. The Cartan subalgebra of diagonal matrices is denoted by $\mathfrak{h}_{m|n}$, which is spanned by $\{e_{ii} | 1 \leq i \leq m+n\}$. We denote by $\{\epsilon_i | 1 \leq i \leq m+n\}$ the basis in $\mathfrak{h}_{m|n}^*$ dual to $\{e_{ii} | 1 \leq i \leq m+n\}$, and the lattice of *integral weights* for $\mathfrak{gl}(m|n)$ by

$$X(m|n) = \sum_{i=1}^{m+n} \mathbb{Z}\epsilon_i.$$

The supertrace form on $\mathfrak{gl}(m|n)$ induces a non-degenerate symmetric bilinear form $(\cdot | \cdot)$ on $\mathfrak{h}_{m|n}^*$ determined by

$$(\epsilon_i | \epsilon_j) = (-1)^{|i|} \delta_{ij}, \quad \text{for } 1 \leq i, j \leq m+n,$$

where we use the notation $|i| := \begin{cases} 0, & \text{if } 1 \leq i \leq m, \\ 1, & \text{if } m+1 \leq i \leq m+n. \end{cases}$ The subalgebra of upper triangular matrices with respect to this standard basis is called the *standard Borel subalgebra* and denoted by \mathfrak{b}_{st} .

In this paper we shall need to deal with various Borel subalgebras of $\mathfrak{gl}(m|n)$ that may not be conjugate to \mathfrak{b}_{st} . For this purpose, let $\mathbf{b} = (b_1, b_2, \dots, b_{m+n})$ be a $0^m 1^n$ -sequence. Such a sequence \mathbf{b} gives rise to a \mathbf{b} -ordered basis $\{e_1^{\mathbf{b}}, e_2^{\mathbf{b}}, \dots, e_{m+n}^{\mathbf{b}}\}$ for $\mathbb{C}^{m|n}$ by rearranging its standard basis as follows: Let $1 \leq i_1 < i_2 < \dots < i_m \leq m+n$ be such that $b_{i_k} = 0$, and $1 \leq j_1 < j_2 < \dots < j_n \leq m+n$ be such that $b_{j_\ell} = 1$. Then

$$e_{i_k}^{\mathbf{b}} = e_k \quad (1 \leq k \leq m), \quad e_{j_\ell}^{\mathbf{b}} = e_{m+\ell} \quad (1 \leq \ell \leq n).$$

For example, for the standard sequence $\mathbf{b}_{\text{st}} = (0, \dots, 0, 1, \dots, 1)$, the \mathbf{b}_{st} -ordered basis is the standard basis, i.e., $e_i^{\mathbf{b}} = e_i$, for $1 \leq i \leq m+n$. On the other hand, if \mathbf{b} consists of n 1's followed by m 0's, then the \mathbf{b} -ordered basis is $\{e_{m+1}, e_{m+2}, \dots, e_{m+n}, e_1, \dots, e_m\}$.

We also realize $\mathfrak{gl}(m|n)$ as $(m+n) \times (m+n)$ matrices with respect to the \mathbf{b} -ordered basis. The (i, j) th elementary matrix here is denoted by $e_{ij}^{\mathbf{b}}$. The Borel subalgebra \mathfrak{b} corresponding to \mathbf{b} (also denoted by $\mathfrak{b}_{\mathbf{b}}$ if necessary) is the subalgebra generated by $e_{ij}^{\mathbf{b}}$ for $1 \leq i \leq j \leq m+n$. The algebras \mathfrak{b} 's for different \mathbf{b} 's are non-conjugate under the (even) group $G_{\bar{0}}$, and the corresponding simple systems associated to different \mathbf{b} 's are representatives among all simple systems for $(\mathfrak{gl}(m|n), \mathfrak{h}_{m|n})$ under the conjugation by its Weyl group $W = \mathfrak{S}_m \times \mathfrak{S}_n$.

The Cartan subalgebras of \mathfrak{b} consisting of diagonal matrices are all equal to $\mathfrak{h}_{m|n}$ (independent of \mathbf{b}). Let $\epsilon_i^{\mathbf{b}} \in \mathfrak{h}_{m|n}^*$ be defined by

$$\langle \epsilon_i^{\mathbf{b}}, e_{jj}^{\mathbf{b}} \rangle = \delta_{ij}, \quad \text{for } 1 \leq i, j \leq m+n.$$

We have

$$\langle \epsilon_i^{\mathbf{b}} | \epsilon_j^{\mathbf{b}} \rangle = (-1)^{b_i} \delta_{ij}, \quad \text{for } 1 \leq i, j \leq m+n.$$

The simple system with respect to the Borel subalgebra \mathfrak{b} associated to \mathbf{b} is

$$\Pi(\mathbf{b}) := \{\epsilon_i^{\mathbf{b}} - \epsilon_{i+1}^{\mathbf{b}} | 1 \leq i \leq m+n-1\},$$

where the parity of $\epsilon_i^{\mathbf{b}} - \epsilon_{i+1}^{\mathbf{b}}$ is $\bar{0}$, if $b_i = b_{i+1}$, and $\bar{1}$ otherwise. Let $\Phi_{\mathbf{b}, \bar{0}}^+$ and $\Phi_{\mathbf{b}, \bar{1}}^+$ be the corresponding sets of positive even and positive odd roots of \mathfrak{b} .

Let $\lambda \in \mathfrak{h}_{m|n}^*$. Fix a $0^m 1^n$ -sequence \mathbf{b} and hence a Borel subalgebra \mathfrak{b} . Let \mathbb{C}_λ be the one-dimensional $\mathfrak{h}_{m|n}$ -module that transforms by λ , which is extended to a \mathfrak{b} -module by letting $e_{ij}^{\mathbf{b}}$ act trivially, for $i < j$. The \mathbf{b} -Verma module of highest weight λ is defined to be

$$M_{\mathbf{b}}(\lambda) := \text{Ind}_{\mathfrak{b}}^{\mathfrak{gl}(m|n)} \mathbb{C}_\lambda,$$

and as usual it has a unique irreducible quotient $\mathfrak{gl}(m|n)$ -module, denoted by $L_{\mathbf{b}}(\lambda)$. We denote by $\text{ch} M$ the (formal) character of a $\mathfrak{gl}(m|n)$ -weight module M as usual. We have the following character formula of the \mathbf{b} -Verma module:

$$\text{ch} M_{\mathbf{b}}(\lambda) = e^\lambda \frac{\prod_{\gamma \in \Phi_{\mathbf{b}, \bar{1}}^+} (1 + e^{-\gamma})}{\prod_{\beta \in \Phi_{\mathbf{b}, \bar{0}}^+} (1 - e^{-\beta})}.$$

6.2. Odd reflection. We follow the notation in §5.2. Take a $0^m 1^n$ -sequence of the form $\mathbf{b} = (\mathbf{b}^1, 0, 1, \mathbf{b}^2)$, where \mathbf{b}^1 and \mathbf{b}^2 are $0^{m_1} 1^{n_1}$ - and $0^{m_2} 1^{n_2}$ -sequences satisfying $m = m_1 + m_2 + 1$ and $n = n_1 + n_2 + 1$. Recall from (5.3) that $\kappa = m_1 + n_1 + 1$. Take the $0^m 1^n$ -sequence $\mathbf{b}' = (\mathbf{b}^1, 1, 0, \mathbf{b}^2)$, *adjacent* to the sequence \mathbf{b} .

Note the simple system $\Pi(\mathbf{b}')$ is obtained from $\Pi(\mathbf{b})$ by an odd reflection with respect to the odd simple root $\alpha = \epsilon_\kappa^{\mathbf{b}} - \epsilon_{\kappa+1}^{\mathbf{b}}$. The corresponding positive systems are related by $\Phi_{\mathbf{b}'}^+ = \Phi_{\mathbf{b}}^+ \cup \{-\alpha\} \setminus \{\alpha\}$.

Lemma 6.1. *Let \mathbf{b}, \mathbf{b}' be adjacent $0^m 1^n$ -sequences as above. Let $\alpha = \epsilon_\kappa^{\mathbf{b}} - \epsilon_{\kappa+1}^{\mathbf{b}}$. Then*

$$\text{ch } M_{\mathbf{b}}(\lambda) = \text{ch } M_{\mathbf{b}'}(\lambda - \alpha).$$

Proof. Let $\Psi := \Phi_{\mathbf{b}, \bar{1}}^+ \setminus \{\alpha\}$. Then $\Phi_{\mathbf{b}', \bar{1}}^+ = \Psi \cup \{-\alpha\}$. Also $\Phi_{\mathbf{b}, \bar{0}}^+ = \Phi_{\mathbf{b}', \bar{0}}^+$. Thus, we have

$$\begin{aligned} \text{ch } M_{\mathbf{b}}(\lambda) &= e^\lambda \frac{\prod_{\gamma \in \Phi_{\mathbf{b}, \bar{1}}^+} (1 + e^{-\gamma})}{\prod_{\beta \in \Phi_{\mathbf{b}, \bar{0}}^+} (1 - e^{-\beta})} = e^\lambda (1 + e^{-\alpha}) \frac{\prod_{\gamma \in \Psi} (1 + e^{-\gamma})}{\prod_{\beta \in \Phi_{\mathbf{b}, \bar{0}}^+} (1 - e^{-\beta})} \\ &= e^{\lambda - \alpha} \frac{\prod_{\gamma \in \Phi_{\mathbf{b}', \bar{1}}^+} (1 + e^{-\gamma})}{\prod_{\beta \in \Phi_{\mathbf{b}, \bar{0}}^+} (1 - e^{-\beta})} = \text{ch } M_{\mathbf{b}'}(\lambda - \alpha). \end{aligned}$$

This proves the lemma. \square

For $\alpha = \epsilon_\kappa^{\mathbf{b}} - \epsilon_{\kappa+1}^{\mathbf{b}}$, we introduce the following notation:

$$(6.1) \quad \lambda^{\mathbb{L}} = \begin{cases} \lambda, & \text{if } (\lambda, \alpha) = 0, \\ \lambda - \alpha, & \text{if } (\lambda, \alpha) \neq 0, \end{cases} \quad \text{for } \lambda \in X(m|n).$$

The following odd reflection lemma is well known (see e.g. [PS, KW]). For completeness, we include a short proof.

Lemma 6.2. *Let \mathbf{b}, \mathbf{b}' be two adjacent $0^m 1^n$ -sequences as above and let $\alpha = \epsilon_\kappa^{\mathbf{b}} - \epsilon_{\kappa+1}^{\mathbf{b}}$. Then $L_{\mathbf{b}}(\lambda) = L_{\mathbf{b}'}(\lambda^{\mathbb{L}})$.*

Proof. Let $\{e_\alpha, e_{-\alpha}, h_\alpha\}$ be the Chevalley generators associated with α . Let v_λ be a nonzero $\mathfrak{b}_{\mathbf{b}}$ -highest weight vector of $L_{\mathbf{b}}(\lambda)$. We consider the two cases separately.

First suppose that $(\lambda, \alpha) = 0$. Then by irreducibility of $L_{\mathbf{b}}(\lambda)$ we have $e_{-\alpha} v_\lambda = 0$. Thus, v_λ is a $\mathfrak{b}_{\mathbf{b}'}$ -singular vector as well, and so $L_{\mathbf{b}}(\lambda) = L_{\mathbf{b}'}(\lambda)$.

Now suppose that $(\lambda, \alpha) \neq 0$. We have $e_\alpha e_{-\alpha} v_\lambda = h_\alpha v_\lambda \neq 0$ in $L_{\mathbf{b}}(\lambda)$, and so $e_{-\alpha} v_\lambda \neq 0$. Now observe that $e_{-\alpha}(e_{-\alpha} v_\lambda) = e_{-\alpha}^2 v_\lambda = 0$, and also, for any $\beta \in \Phi_{\mathbf{b}}^+ \setminus \{\alpha\}$, if $\beta - \alpha$ is a root, then $\beta - \alpha \in \Phi_{\mathbf{b}}^+$. It follows that, for any root vector X_β corresponding to such that a β , we have $X_\beta e_{-\alpha} v_\lambda = 0$. Thus, $e_{-\alpha} v_\lambda$ is a $\mathfrak{b}_{\mathbf{b}'}$ -singular vector and hence $L_{\mathbf{b}}(\lambda) = L_{\mathbf{b}'}(\lambda - \alpha)$. \square

6.3. BGG category. For $\mu \in \mathfrak{h}_{m|n}^*$ and a $\mathfrak{gl}(m|n)$ -module M we denote the μ -weight space of M as usual by $M_\mu = \{x \in M \mid hx = \mu(h)x, \forall h \in \mathfrak{h}_{m|n}\}$.

Definition 6.3. Let \mathbf{b} be a $0^m 1^n$ -sequence. The Bernstein-Gelfand-Gelfand (BGG) category $\mathcal{O}_{\mathbf{b}}^{m|n}$ is the category of finitely generated $\mathfrak{h}_{m|n}$ -semisimple $\mathfrak{gl}(m|n)$ -modules M such that

- (i) $M = \bigoplus_{\mu \in X(m|n)} M_\mu$ and $\dim M_\mu < \infty$;
- (ii) there exist finitely many weights ${}^1\lambda, {}^2\lambda, \dots, {}^k\lambda \in X(m|n)$ (depending on M) such that if μ is a weight in M , then $\mu \in {}^i\lambda - \sum_{\alpha \in \Pi(\mathfrak{b})} \mathbb{Z}_+\alpha$, for some i .

The morphisms in $\mathcal{O}_{\mathfrak{b}}^{m|n}$ are all (not necessarily even) homomorphisms of $\mathfrak{gl}(m|n)$ -modules.

In short, $\mathcal{O}_{\mathfrak{b}}^{m|n}$ is the category of finitely generated integral weight $\mathfrak{gl}(m|n)$ -modules that are \mathfrak{b} -locally finite. The $\mathfrak{gl}(m|n)$ -modules $L_{\mathfrak{b}}(\lambda)$ and $M_{\mathfrak{b}}(\lambda)$, for $\lambda \in X(m|n)$, are objects in the BGG category $\mathcal{O}_{\mathfrak{b}}^{m|n}$.

Let $M \in \mathcal{O}_{\mathfrak{b}}^{m|n}$ so that $M = \bigoplus_{\gamma \in X(m|n)} M_\gamma$. For $\vartheta \in \mathbb{Z}_2$ we define

$$X(m|n)_{\vartheta} := \left\{ \gamma \in X(m|n) \mid \sum_{i=m+1}^n \langle \gamma, e_{ii} \rangle \equiv \vartheta \right\}.$$

Introduce the following $\mathfrak{gl}(m|n)$ -module whose \mathbb{Z}_2 -grading is specified by the action of $\mathfrak{gl}(m|n)$ (cf. [CL])

$$(6.2) \quad M' = M'_0 \oplus M'_1, \quad \text{where } M'_{\vartheta} := \bigoplus_{\gamma \in X(m|n)_{\vartheta}} M_{\gamma} \quad (\vartheta \in \mathbb{Z}_2).$$

Then $M' \in \mathcal{O}_{\mathfrak{b}}^{m|n}$, and by definition $M \cong M'$ in $\mathcal{O}_{\mathfrak{b}}^{m|n}$. Let us denote by $\mathcal{O}_{\mathfrak{b},0}^{m|n}$ the full subcategory of $\mathcal{O}_{\mathfrak{b}}^{m|n}$ consisting of objects with \mathbb{Z}_2 -gradation given by (6.2). Then all morphisms in $\mathcal{O}_{\mathfrak{b},0}^{m|n}$ are automatically even, and hence $\mathcal{O}_{\mathfrak{b},0}^{m|n}$ is an abelian category. Since the categories $\mathcal{O}_{\mathfrak{b},0}^{m|n}$ and $\mathcal{O}_{\mathfrak{b}}^{m|n}$ have isomorphic skeleton subcategories, $\mathcal{O}_{\mathfrak{b},0}^{m|n}$ and $\mathcal{O}_{\mathfrak{b}}^{m|n}$ are equivalent categories. It follows that $\mathcal{O}_{\mathfrak{b}}^{m|n}$ is an abelian category.

We adopt the following convention. When dealing with the BGG category associated to the standard $0^m 1^n$ -sequence \mathfrak{b}_{st} and the standard Borel subalgebra \mathfrak{b}_{st} , we will drop the subscript \mathfrak{b}_{st} and write the corresponding category, Verma module, and irreducible module as $\mathcal{O}^{m|n}$, $M(\lambda)$, and $L(\lambda)$, respectively.

Proposition 6.4. *The categories $\mathcal{O}_{\mathfrak{b}}^{m|n}$ are identical, for all $0^m 1^n$ -sequences \mathfrak{b} .*

Proof. We shall show that the category $\mathcal{O}_{\mathfrak{b}}^{m|n}$ for a fixed \mathfrak{b} is identical to $\mathcal{O}^{m|n}$.

It is clear that any two $0^m 1^n$ -sequences, say \mathfrak{b} and \mathfrak{b}_{st} , can be connected via a sequence of $0^m 1^n$ -sequences such that any two neighboring sequences are adjacent. Accordingly, the Borel subalgebras \mathfrak{b} and \mathfrak{b}_{st} can be converted to one another via a sequence of odd reflections. It follows by this observation and Lemma 6.2 that the categories $\mathcal{O}_{\mathfrak{b}}^{m|n}$ and $\mathcal{O}^{m|n}$ have the same collection of simple objects, denoted by $\text{Irr}\mathcal{O}$.

Since every Verma module $M(\lambda) \in \mathcal{O}^{m|n}$ has finite length when regarded as a $\mathfrak{gl}(m|n)_{\bar{0}}$ -module, it has finite length as a $\mathfrak{gl}(m|n)$ -module as well. It follows by this and the character comparisons in Lemmas 6.1 and 6.2 that every \mathfrak{b}' -Verma module $M_{\mathfrak{b}}(\lambda) \in \mathcal{O}_{\mathfrak{b}}^{m|n}$ has finite length with composition factors in $\text{Irr}\mathcal{O}$. So we conclude that both categories $\mathcal{O}^{m|n}$ and $\mathcal{O}_{\mathfrak{b}}^{m|n}$ can be characterized as the category of integral weight

$\mathfrak{h}_{m|n}$ -semisimple $\mathfrak{gl}(m|n)$ -modules that have finite composition series with composition factors in $\text{Irr}\mathcal{O}$, and hence $\mathcal{O}_{\mathbf{b}}^{m|n} = \mathcal{O}^{m|n}$. \square

6.4. Weyl vectors. The supertrace function

$$(6.3) \quad \text{Str} = \sum_{i=1}^m \epsilon_i - \sum_{j=1}^n \epsilon_{m+j}$$

satisfies the fundamental property that

$$(6.4) \quad (\text{Str}|\gamma) = 0, \quad \forall \gamma \in \Phi.$$

Let \mathfrak{b} be the Borel subalgebra corresponding to the $0^m 1^n$ -sequence \mathbf{b} . Recall $\Phi_{\mathbf{b},0}^+$ and $\Phi_{\mathbf{b},1}^+$ denote the sets of positive even and positive odd roots of \mathfrak{b} , respectively. We define the Weyl vector $\tilde{\rho}_{\mathbf{b}}$ and its normalized version $\rho_{\mathbf{b}}$ by

$$(6.5) \quad \begin{aligned} \tilde{\rho}_{\mathbf{b}} &:= \frac{1}{2} \sum_{\alpha \in \Phi_{\mathbf{b},0}^+} \alpha - \frac{1}{2} \sum_{\beta \in \Phi_{\mathbf{b},1}^+} \beta, \\ \rho_{\mathbf{b}} &:= \tilde{\rho}_{\mathbf{b}} + \frac{m-n+1}{2} \text{Str}. \end{aligned}$$

We shall always use the normalized $\rho_{\mathbf{b}}$, which behaves as well as $\tilde{\rho}_{\mathbf{b}}$ in most circumstances and is more convenient for our purpose.

Lemma 6.5. *The element $\rho_{\mathbf{b}} \in \mathfrak{h}_{m|n}^*$ is characterized by the following two properties:*

- (i) $(\rho_{\mathbf{b}}|\beta) = \frac{1}{2}(\beta|\beta)$, for every simple root $\beta \in \Pi(\mathbf{b})$.
- (ii) $(\rho_{\mathbf{b}}|\epsilon_{m+n}) = \begin{cases} 0, & \text{if } b_{m+n} = 1, \\ 1, & \text{if } b_{m+n} = 0. \end{cases}$

Moreover, we have $\rho_{\mathbf{b}} \in X(m|n)$, for every \mathbf{b} .

Lemma 6.5 implies that $\rho_{\mathbf{b}}$ here coincides with the one used by Kujawa [Ku, 2.7].

Proof. Clearly an element satisfying (i) and (ii) is unique. Thanks to (6.4), we have $(\rho_{\mathbf{b}}|\beta) = (\tilde{\rho}_{\mathbf{b}}|\beta) = \frac{1}{2}(\beta|\beta)$, for every simple root $\beta \in \Pi(\mathbf{b})$. So it remains to show that $\rho_{\mathbf{b}}$ defined in (6.5) satisfies (ii).

A direct computation shows that, for $\mathbf{b}_{\text{st}} = (0, \dots, 0, 1, \dots, 1)$,

$$\rho_{\mathbf{b}_{\text{st}}} = \sum_{i=1}^m (m+1-i-n)\epsilon_i + \sum_{j=m+1}^{m+n} (m+n-j)\epsilon_j,$$

and so $\rho_{\mathbf{b}_{\text{st}}}$ satisfies (ii). As observed in proof of Proposition 6.4, the Borel subalgebra \mathfrak{b} (associated to \mathbf{b}) and \mathfrak{b}_{st} can be converted to one another via a sequence of odd reflections. So it remains to verify the following consistency of (ii): if the property (ii) holds for $\mathbf{b} = (\mathbf{b}^1, 0, 1, \mathbf{b}^2)$ (with 0 and 1 at κ th and $(\kappa+1)$ th places) then it holds for the adjacent sequence $\mathbf{b}' = (\mathbf{b}^1, 1, 0, \mathbf{b}^2)$. Note that $\rho_{\mathbf{b}'} = \rho_{\mathbf{b}} + \alpha$, where $\alpha = \epsilon_{\kappa}^{\mathbf{b}} - \epsilon_{\kappa+1}^{\mathbf{b}}$. Now this consistency of (ii) follows from by a quick case-by-case checking, depending on whether or not \mathbf{b}^2 is empty. \square

Define a bijection

$$(6.6) \quad X(m|n) \longrightarrow \mathbb{Z}^{m+n}, \quad \lambda \mapsto f_\lambda^{\mathbf{b}},$$

where $f_\lambda^{\mathbf{b}} \in \mathbb{Z}^{m+n}$ is defined by letting

$$(6.7) \quad f_\lambda^{\mathbf{b}}(i) := (\lambda + \rho_{\mathbf{b}} | \epsilon_i^{\mathbf{b}}), \quad \forall i \in [m+n].$$

The *Bruhat ordering* with respect to the Borel subalgebra \mathfrak{b} is the partial ordering on $X(m|n)$ induced by the Bruhat ordering $\succeq_{\mathbf{b}}$ on \mathbb{Z}^{m+n} under the above bijection. This terminology is justified by the role it plays in representation theory of $\mathfrak{gl}(m|n)$ (see [CW2, Section 2.2]; also see [Br1, Se] in the case of the standard Borel \mathfrak{b}_{st}). The Bruhat ordering on $X(m|n)$ will also be denoted by $\succeq_{\mathbf{b}}$ by abuse of notation. Recall d_i from (2.5). For adjacent sequences \mathbf{b} and \mathbf{b}' (for notations see §5.2 or the above proof of Lemma 6.5), we have

$$f_\lambda^{\mathbf{b}'} = f_\lambda^{\mathbf{b}} + d_\kappa - d_{\kappa+1}, \quad \forall \lambda \in X(m|n).$$

Now consider the standard sequence \mathbf{b}_{st} and the standard Borel subalgebra \mathfrak{b}_{st} . A weight $\lambda \in X(m|n)$ is called *typical* if $f_\lambda^{\mathbf{b}_{\text{st}}}(i) \neq f_\lambda^{\mathbf{b}_{\text{st}}}(j)$ for all i, j such that $1 \leq i \leq m < j \leq m+n$, and λ is *anti-dominant* if $f_\lambda^{\mathbf{b}_{\text{st}}}(1) \leq f_\lambda^{\mathbf{b}_{\text{st}}}(2) \leq \dots \leq f_\lambda^{\mathbf{b}_{\text{st}}}(m)$ and $f_\lambda^{\mathbf{b}_{\text{st}}}(m+1) \geq \dots \geq f_\lambda^{\mathbf{b}_{\text{st}}}(m+n)$.

6.5. Tilting modules. Recall that the Lie superalgebra $\mathfrak{gl}(m|n)$ has an automorphism τ given by the formula:

$$\tau(e_{ij}) := -(-1)^{|i|(|i|+|j|)} e_{ji}.$$

For an object $M = \oplus_{\nu \in X(m|n)} M_\nu \in \mathcal{O}^{m|n}$, we let

$$M^\vee := \oplus_{\nu \in X(m|n)} M_\nu^*$$

be the restricted dual of M . We may define an action of $\mathfrak{gl}(m|n)$ on M^\vee by $(g \cdot f)(x) := -f(\tau(g)x)$, for $f \in M^\vee$, $g \in \mathfrak{gl}(m|n)$, and $x \in M$. We denote the resulting $\mathfrak{gl}(m|n)$ -module by M^τ , which is an object in $\mathcal{O}^{m|n}$. An object $M \in \mathcal{O}^{m|n}$ is called *self-dual*, if $M^\tau \cong M$. Clearly, $L(\lambda)$ is self-dual, for all $\lambda \in X(m|n)$.

Fix an arbitrary $0^m 1^n$ -sequence \mathbf{b} . An object $M \in \mathcal{O}^{m|n}$ is said to have a \mathbf{b} -Verma flag (respectively, a *dual \mathbf{b} -Verma flag*), if M has a filtration

$$M_0 = 0 \subseteq M_1 \subseteq M_2 \subseteq \dots \subseteq M_t = M,$$

such that $M_i/M_{i-1} \cong M_{\mathbf{b}}(\gamma_i)$ (respectively, $M_i/M_{i-1} \cong M_{\mathbf{b}}(\gamma_i)^\tau$), for some $\gamma_i \in X(m|n)$ and $1 \leq i \leq t$.

Definition 6.6. Associated with each $\lambda \in X(m|n)$, a \mathbf{b} -tilting module $T_{\mathbf{b}}(\lambda)$ is an indecomposable $\mathfrak{gl}(m|n)$ -module in $\mathcal{O}^{m|n}$ satisfying the following two conditions:

- (i) $T_{\mathbf{b}}(\lambda)$ has a \mathbf{b} -Verma flag with $M_{\mathbf{b}}(\lambda)$ at the bottom.
- (ii) $\text{Ext}_{\mathcal{O}^{m|n}}^1(M_{\mathbf{b}}(\mu), T_{\mathbf{b}}(\lambda)) = 0$, for all $\mu \in X(m|n)$.

Combining [Br2, Theorem 6.3, Lemma 7.3] with [Br1, Theorem 6.4], as a super generalization of [So2], we conclude that the \mathbf{b} -tilting module $T_{\mathbf{b}}(\lambda)$, for every $\lambda \in X(m|n)$, in the category $\mathcal{O}^{m|n}$ exists and is unique (nevertheless, it depends on \mathbf{b}).

Let $\mathcal{O}_{\mathbf{b}}^{m|n,\Delta}$ denote the full subcategory of $\mathcal{O}^{m|n}$ consisting of objects that have finite \mathbf{b} -Verma flags.

The following lemma is standard in a highest weight category [Don] (for a proof see e.g. [CW1, Proposition 3.7]).

Lemma 6.7. *Let \mathbf{b} be a $0^m 1^n$ -sequence.*

- (i) *If $N \in \mathcal{O}^{m|n}$ has a \mathbf{b} -Verma flag, then $\text{Ext}_{\mathcal{O}^{m|n}}^1(N, M_{\mathbf{b}}(\mu)^\tau) = 0$, for all $\mu \in X(m|n)$.*
- (ii) *$N \in \mathcal{O}^{m|n}$ has a dual \mathbf{b} -Verma flag if and only if $\text{Ext}_{\mathcal{O}^{m|n}}^1(M_{\mathbf{b}}(\mu), N) = 0$, for all $\mu \in X(m|n)$.*

We have the following useful characterization of tilting modules, which is well known in the algebraic group or Kac-Moody setting (cf. [Don, So1]). The same proof can be adapted in our setting, using Lemma 6.7 and [So2, Proposition 5.6].

Lemma 6.8. *A $\mathfrak{gl}(m|n)$ -module $T \in \mathcal{O}^{m|n}$ is a \mathbf{b} -tilting module if and only if T is an indecomposable self-dual module that has a \mathbf{b} -Verma flag.*

When dealing with the standard $0^m 1^n$ -sequence \mathbf{b}_{st} and the standard Borel \mathbf{b}_{st} , we shall continue the convention of suppressing \mathbf{b}_{st} , and hence denote the \mathbf{b}_{st} -tilting modules by $T(\lambda)$.

Proposition 6.9. *Let $T(\lambda)$ be the tilting module corresponding to $\lambda \in X(m|n)$ in $\mathcal{O}^{m|n}$. Then $T(\lambda)$ is a \mathbf{b} -tilting module, for an arbitrary $0^m 1^n$ -sequence \mathbf{b} .*

Proof. Fix an arbitrary $0^m 1^n$ -sequence \mathbf{b} .

Let $\mu \in X(m|n)$ be anti-dominant and typical. Then it is well known that (cf. e.g. [Kac2, Se, Br1]) the Verma module $M(\mu)$ with respect to the standard Borel \mathbf{b}_{st} is irreducible, and hence $M(\mu)^\tau \cong M(\mu)$. This implies that $L(\mu) = M(\mu) = T(\mu)$. Hence, $M(\mu)$ is equal to a \mathbf{b} -Verma module $M_{\mathbf{b}}(\mu^{\mathbf{b}})$, for some $\mu^{\mathbf{b}} \in X(m|n)$.

Now fix $\lambda \in X(m|n)$. Then it is easy to find an anti-dominant typical weight $\mu \in X(m|n)$ and a weight $\gamma \in X(m|n)$ such that $\dim_{\mathbb{C}} L(\gamma) < \infty$ and $\lambda = \mu + \gamma$. The $\mathfrak{gl}(m|n)$ -module $M(\mu) \otimes L(\gamma)$ is self-dual (as a tensor product of two simples) and has a \mathbf{b}_{st} -Verma flag, in which $M(\lambda)$ appears as a subquotient exactly once. By some standard argument which goes back to Soergel, any direct summand of $M(\mu) \otimes L(\gamma)$ is self-dual and also has a \mathbf{b}_{st} -Verma flag. The unique summand T containing $M(\lambda)$ must have $M(\lambda)$ at the bottom, since λ is the highest weight in $M(\mu) \otimes L(\gamma)$. Hence we have $T \cong T(\lambda)$.

Since $M(\mu) = M_{\mathbf{b}}(\mu^{\mathbf{b}})$, the tensor product $M(\mu) \otimes L(\gamma)$ also has a \mathbf{b} -Verma flag. Now the indecomposable summand T has a \mathbf{b} -Verma flag and is also self-dual. Hence, it must be a \mathbf{b} -tilting module by Lemma 6.8. \square

Let \mathbf{b} and \mathbf{b}' be two adjacent $0^m 1^n$ -sequences such that the corresponding Borel subalgebra \mathbf{b}' is obtained from \mathbf{b} via the odd reflection with respect to the simple root α of \mathbf{b} . We introduce the following notation:

$$(6.8) \quad \lambda^{\mathbb{U}} = \begin{cases} \lambda - 2\alpha, & \text{if } (\lambda, \alpha) = 0, \\ \lambda - \alpha, & \text{if } (\lambda, \alpha) \neq 0, \end{cases} \quad \text{for } \lambda \in X(m|n).$$

The following may be regarded as a “dual version” of Lemma 6.2.

Theorem 6.10. *Let \mathbf{b} and \mathbf{b}' be two adjacent $0^m 1^n$ -sequences such that the Borel subalgebra \mathfrak{b}' is obtained from \mathfrak{b} via the odd reflection with respect to the simple root α of \mathfrak{b} . Then*

$$T_{\mathbf{b}}(\lambda) = T_{\mathbf{b}'}(\lambda^{\mathbb{U}}), \quad \text{for } \lambda \in X(m|n).$$

Proof. By Proposition 6.9, the \mathbf{b} -tilting module $T_{\mathbf{b}}(\lambda)$ is also a \mathbf{b}' -tilting module. Since the \mathbf{b}' -tilting modules form a basis of the Grothendieck group of $\mathcal{O}_{\mathbf{b}'}^{m|n, \Delta}$, in order to prove the theorem, it suffices to prove the following character identities:

$$\text{ch} T_{\mathbf{b}}(\lambda) = \begin{cases} \text{ch} T_{\mathbf{b}'}(\lambda - 2\alpha), & \text{if } (\lambda, \alpha) = 0, \\ \text{ch} T_{\mathbf{b}'}(\lambda - \alpha), & \text{if } (\lambda, \alpha) \neq 0. \end{cases}$$

By Soergel’s character formula for tilting modules [So2, Theorem 6.7] and its super generalization [Br2, Theorem 6.4], we have, for an arbitrary \mathbf{b} ,

$$(6.9) \quad (T_{\mathbf{b}}(\lambda) : M_{\mathbf{b}}(\mu)) = [M_{\mathbf{b}}(-\mu - 2\rho_{\mathbf{b}}) : L_{\mathbf{b}}(-\lambda - 2\rho_{\mathbf{b}})].$$

Using (6.9) we compute

$$\begin{aligned} \text{ch} T_{\mathbf{b}}(\lambda) &= \sum_{\mu} (T_{\mathbf{b}}(\lambda) : M_{\mathbf{b}}(\mu)) \text{ch} M_{\mathbf{b}}(\mu) \\ (6.10) \quad &= \sum_{\mu} [M_{\mathbf{b}}(-\mu - 2\rho_{\mathbf{b}}) : L_{\mathbf{b}}(-\lambda - 2\rho_{\mathbf{b}})] \text{ch} M_{\mathbf{b}}(\mu). \end{aligned}$$

We now apply Lemmas 6.1 and 6.2, and the identity $\rho_{\mathbf{b}'} = \rho_{\mathbf{b}} + \alpha$ in two separate cases. We shall also need (6.9) for varying \mathbf{b} , λ , μ .

Case (i). Assume $(\lambda, \alpha) = 0$. Continuing (6.10), we have

$$\begin{aligned} \text{ch} T_{\mathbf{b}}(\lambda) &= \sum_{\mu} [M_{\mathbf{b}'}(-\mu - 2\rho_{\mathbf{b}} - \alpha) : L_{\mathbf{b}'}(-\lambda - 2\rho_{\mathbf{b}})] \text{ch} M_{\mathbf{b}'}(\mu - \alpha) \\ &= \sum_{\mu} [M_{\mathbf{b}'}(-\mu - 2\rho_{\mathbf{b}'} + \alpha) : L_{\mathbf{b}'}(-\lambda - 2\rho_{\mathbf{b}'} + 2\alpha)] \text{ch} M_{\mathbf{b}'}(\mu - \alpha) \\ &= \sum_{\mu} (T_{\mathbf{b}'}(\lambda - 2\alpha) : M_{\mathbf{b}'}(\mu - \alpha)) \text{ch} M_{\mathbf{b}'}(\mu - \alpha) \\ &= \text{ch} T_{\mathbf{b}'}(\lambda - 2\alpha). \end{aligned}$$

Case (ii). Assume $(\lambda, \alpha) \neq 0$. Continuing (6.10) again, we have

$$\begin{aligned} \text{ch} T_{\mathbf{b}}(\lambda) &= \sum_{\mu} [M_{\mathbf{b}'}(-\mu - 2\rho_{\mathbf{b}} - \alpha) : L_{\mathbf{b}'}(-\lambda - 2\rho_{\mathbf{b}} - \alpha)] \text{ch} M_{\mathbf{b}'}(\mu - \alpha) \\ &= \sum_{\mu} [M_{\mathbf{b}'}(-\mu - 2\rho_{\mathbf{b}'} + \alpha) : L_{\mathbf{b}'}(-\lambda - 2\rho_{\mathbf{b}} + \alpha)] \text{ch} M_{\mathbf{b}'}(\mu - \alpha) \\ &= \sum_{\mu} (T_{\mathbf{b}'}(\lambda - \alpha) : M_{\mathbf{b}'}(\mu - \alpha)) \text{ch} M_{\mathbf{b}'}(\mu - \alpha) \\ &= \text{ch} T_{\mathbf{b}'}(\lambda - \alpha). \end{aligned}$$

This completes the proof. \square

6.6. Auxiliary modules. Let $\mathbf{b} = (b_1, \dots, b_{m+n})$ and \mathbf{b}' be two $0^m 1^n$ -sequences adjacent by the simple odd root $\alpha = \epsilon_{\kappa}^{\mathbf{b}} - \epsilon_{\kappa+1}^{\mathbf{b}}$ of \mathbf{b} as before, and let \mathfrak{b} and \mathfrak{b}' be the corresponding Borel subalgebras again. For definiteness let us assume that $(\epsilon_{\kappa}^{\mathbf{b}}, \epsilon_{\kappa}^{\mathbf{b}}) = 1 = -(\epsilon_{\kappa+1}^{\mathbf{b}}, \epsilon_{\kappa+1}^{\mathbf{b}})$, i.e., $b_{\kappa} = 0, b_{\kappa+1} = 1$.

Let v_{λ} be a \mathbf{b} -highest weight vector of the \mathbf{b} -Verma module $M_{\mathbf{b}}(\lambda)$. We denote by $e_{\pm\alpha}$ the root vectors corresponding to the roots $\pm\alpha$.

Suppose that $(\lambda, \alpha) = 0$. The Lie superalgebra

$$\mathfrak{a}_{\alpha} := \mathfrak{h}_{m|n} + \mathbb{C}e_{\alpha} + \mathbb{C}e_{-\alpha}$$

is isomorphic to a direct sum of $\mathfrak{gl}(1|1)$ and a subalgebra of $\mathfrak{h}_{m|n}$. Thus, the Verma module of \mathfrak{a}_{α} of highest weight λ , denoted by $M_{(b_{\kappa}, b_{\kappa+1})}(\lambda)$, is two-dimensional. The irreducible modules of \mathfrak{a}_{α} of highest weight λ and $\lambda - \alpha$, denoted by \mathbb{C}_{λ} and $\mathbb{C}_{\lambda-\alpha}$, respectively, are one-dimensional and we have the following exact sequence of \mathfrak{a}_{α} -modules

$$(6.11) \quad 0 \longrightarrow \mathbb{C}_{\lambda-\alpha} \longrightarrow M_{(b_{\kappa}, b_{\kappa+1})}(\lambda) \longrightarrow \mathbb{C}_{\lambda} \longrightarrow 0.$$

We denote by \mathfrak{n} the radical corresponding to the Borel subalgebra \mathfrak{b} , and by $\mathfrak{n}_{\neq\alpha}$ the subalgebra of \mathfrak{n} spanned by the root spaces $\mathfrak{gl}(m|n)_{\beta}$, for $\beta \neq \alpha$. We observe that $\mathfrak{a}_{\alpha} + \mathfrak{n}_{\neq\alpha} = \mathfrak{b} + \mathbb{C}e_{-\alpha}$ and it contains $\mathfrak{n}_{\neq\alpha}$ as an ideal. Thus, (6.11) extends trivially to an exact sequence of $(\mathfrak{b} + \mathbb{C}e_{-\alpha})$ -modules.

Noting that $(\lambda, -\alpha) = 0$, we can switch the role of α with $-\alpha$ (and \mathbf{b} with \mathbf{b}' at the same time) above. Regarding \mathbb{C}_{λ} as the one-dimensional $(\mathfrak{b} + \mathbb{C}e_{-\alpha})$ -module or similarly regarding \mathbb{C}_{λ} as the one-dimensional $(\mathfrak{b}' + \mathbb{C}e_{\alpha})$ -module, we may form the parabolic Verma modules

$$N_{\mathbf{b}}(\lambda) := \text{Ind}_{\mathfrak{b} + \mathbb{C}e_{-\alpha}}^{\mathfrak{gl}(m|n)} \mathbb{C}_{\lambda}, \quad N_{\mathbf{b}'}(\lambda) := \text{Ind}_{\mathfrak{b}' + \mathbb{C}e_{\alpha}}^{\mathfrak{gl}(m|n)} \mathbb{C}_{\lambda}$$

Observing $\mathfrak{b}' + \mathbb{C}e_{\alpha} = \mathfrak{b} + \mathbb{C}e_{-\alpha}$, we have $N_{\mathbf{b}}(\lambda) = N_{\mathbf{b}'}(\lambda)$.

We continue to assume that $(\lambda, \alpha) = 0$. The tilting \mathfrak{a}_{α} -module of highest weight λ will be denoted by $T_{(b_{\kappa}, b_{\kappa+1})}(\lambda)$. We have the following exact sequence of \mathfrak{a}_{α} -modules (see [Br1, Theorem 4.37 for $m = n = 1$] and compare with Lemma 5.1):

$$(6.12) \quad 0 \longrightarrow M_{(b_{\kappa}, b_{\kappa+1})}(\lambda) \longrightarrow T_{(b_{\kappa}, b_{\kappa+1})}(\lambda) \longrightarrow M_{(b_{\kappa}, b_{\kappa+1})}(\lambda - \alpha) \longrightarrow 0,$$

As before, (6.12) may be regarded as an exact sequence of $(\mathfrak{b} + \mathbb{C}e_{-\alpha})$ -modules with trivial action by $\mathfrak{n}_{\neq\alpha}$. We form the $\mathfrak{gl}(m|n)$ -module

$$U_{\mathbf{b}}(\lambda) := \text{Ind}_{\mathfrak{b} + \mathbb{C}e_{-\alpha}}^{\mathfrak{gl}(m|n)} T_{(b_{\kappa}, b_{\kappa+1})}(\lambda).$$

We can similarly form the module $U_{\mathbf{b}'}(\lambda) := \text{Ind}_{\mathfrak{b}' + \mathbb{C}e_{\alpha}}^{\mathfrak{gl}(m|n)} T_{(b_{\kappa+1}, b_{\kappa})}(\lambda)$. We note that

$$U_{\mathbf{b}}(\lambda) = U_{\mathbf{b}'}(\lambda - 2\alpha)$$

since $T_{(b_{\kappa}, b_{\kappa+1})}(\lambda) = T_{(b_{\kappa+1}, b_{\kappa})}(\lambda - 2\alpha)$ by Theorem 6.10.

In the case that $(\lambda, \alpha) \neq 0$, we define $U_{\mathbf{b}}(\lambda), N_{\mathbf{b}}(\lambda), U_{\mathbf{b}'}(\lambda)$ and $N_{\mathbf{b}'}(\lambda)$ to be

$$(6.13) \quad U_{\mathbf{b}}(\lambda) = N_{\mathbf{b}}(\lambda) = M_{\mathbf{b}}(\lambda), \quad U_{\mathbf{b}'}(\lambda) = N_{\mathbf{b}'}(\lambda) = M_{\mathbf{b}'}(\lambda).$$

Recall the notation $\lambda^{\mathbb{L}}$ from (6.1) and $\lambda^{\mathbb{U}}$ from (6.8). Summarizing the above two cases and using Lemma 6.1, we have

$$(6.14) \quad \text{ch } N_{\mathbf{b}}(\lambda) = \text{ch } N_{\mathbf{b}'}(\lambda^{\mathbb{L}}), \quad \text{ch } U_{\mathbf{b}}(\lambda) = \text{ch } U_{\mathbf{b}'}(\lambda^{\mathbb{U}}), \quad \text{for } \lambda \in X(m|n).$$

Lemma 6.11. *Let \mathbf{b} be a $0^m 1^n$ -sequence and $\alpha = \epsilon_{\kappa}^{\mathbf{b}} - \epsilon_{\kappa+1}^{\mathbf{b}}$ be an isotropic simple root as above. Let $\lambda \in X(m|n)$ be such that $(\lambda, \alpha) = 0$. Then we have the following short exact sequences of $\mathfrak{gl}(m|n)$ -modules:*

$$\begin{aligned} 0 &\longrightarrow N_{\mathbf{b}}(\lambda - \alpha) \longrightarrow M_{\mathbf{b}}(\lambda) \longrightarrow N_{\mathbf{b}}(\lambda) \longrightarrow 0, \\ 0 &\longrightarrow M_{\mathbf{b}}(\lambda) \longrightarrow U_{\mathbf{b}}(\lambda) \longrightarrow M_{\mathbf{b}}(\lambda - \alpha) \longrightarrow 0. \end{aligned}$$

Proof. The first exact sequence is obtained by regarding the exact sequence (6.11) as an exact sequence of $(\mathfrak{b} + \mathbb{C}e_{-\alpha})$ -modules, and then inducing it to an exact sequence of $\mathfrak{gl}(m|n)$ -modules. The second exact sequence is obtained similarly, now using (6.12) in place of (6.11). \square

By (6.13) and Lemma 6.11, we have

$$(6.15) \quad \text{ch } M_{\mathbf{b}}(\lambda) = \begin{cases} \text{ch } N_{\mathbf{b}}(\lambda) + \text{ch } N_{\mathbf{b}}(\lambda - \alpha), & \text{if } (\lambda, \alpha) = 0, \\ \text{ch } N_{\mathbf{b}}(\lambda), & \text{if } (\lambda, \alpha) \neq 0; \end{cases}$$

$$(6.16) \quad \text{ch } U_{\mathbf{b}}(\lambda) = \begin{cases} \text{ch } M_{\mathbf{b}}(\lambda) + \text{ch } M_{\mathbf{b}}(\lambda - \alpha), & \text{if } (\lambda, \alpha) = 0, \\ \text{ch } M_{\mathbf{b}}(\lambda), & \text{if } (\lambda, \alpha) \neq 0. \end{cases}$$

Remark 6.12. Similarly, we have a short exact sequence of $\mathfrak{gl}(m|n)$ -modules:

$$0 \longrightarrow N_{\mathbf{b}'}(\lambda) \longrightarrow M_{\mathbf{b}'}(\lambda - \alpha) \longrightarrow N_{\mathbf{b}'}(\lambda - \alpha) \longrightarrow 0.$$

Since $N_{\mathbf{b}}(\lambda) = N_{\mathbf{b}'}(\lambda)$ and $N_{\mathbf{b}}(\lambda - \alpha) = N_{\mathbf{b}'}(\lambda - \alpha)$, we see that $M_{\mathbf{b}}(\lambda)$ and $M_{\mathbf{b}'}(\lambda - \alpha)$ are opposite extension of two modules.

Remark 6.13. All the results in Section 6 remain valid for an arbitrary basic Lie superalgebra, such as $\mathfrak{osp}(m|2n)$ (cf. [CW2]). This, in particular, applies to Propositions 6.4 and 6.9, Theorem 6.10, and Lemma 6.11.

7. SUPER DUALITY FOR GENERAL LINEAR LIE SUPERALGEBRAS

In this section, we establish a super duality, which is a certain equivalence of categories and identification of Kazhdan-Lusztig theories. In contrast to earlier formulations by the authors, we allow the head (Dynkin) diagrams to correspond to Lie superalgebras. The equivalence established here will be needed for an inductive argument in the proof of Brundan's conjecture next section.

7.1. Infinite-rank Lie superalgebras. Define the sets

$$\begin{aligned} \widetilde{\mathbb{I}} &:= \left\{ 1, 2, \dots, m+n; \frac{1}{2}, \underline{1}, \frac{3}{2}, \dots \right\}, \\ \mathbb{I} &:= \{ 1, 2, \dots, m+n; \underline{1}, \underline{2}, \underline{3}, \dots \}, \\ \check{\mathbb{I}} &:= \left\{ 1, 2, \dots, m+n; \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots \right\}. \end{aligned}$$

Let $\mathbf{b} = (b_1, b_2, \dots, b_{m+n})$ be a $0^m 1^n$ -sequence.

Let \widetilde{V} denote the complex vector superspace with homogeneous ordered basis $\{e_i^{\mathbf{b}} | 1 \leq i \leq m+n\} \cup \{e_{\underline{r}} | r \in \frac{1}{2}\mathbb{N}\}$. Recall that the \mathbb{Z}_2 -gradation of $e_i^{\mathbf{b}}$ is given by $|e_i^{\mathbf{b}}| = b_i$.

The \mathbb{Z}_2 -gradation for the e_r 's is defined by $|e_r| = \overline{2r} \in \mathbb{Z}_2$. We denote by $\tilde{\mathfrak{g}}$ the Lie superalgebra of endomorphisms of \tilde{V} vanishing on all but finitely many e_r 's, $r \in \tilde{\mathbb{I}}$. For $r, s, p \in \tilde{\mathbb{I}}$, let E_{rs} denote the endomorphism defined by $E_{rs}(e_p) := \delta_{sp}e_r$. Then $\tilde{\mathfrak{g}}$ has a basis given by $\{E_{rs} | r, s \in \tilde{\mathbb{I}}\}$. The subalgebra spanned by $\{E_{ij} | 1 \leq i, j \leq m+n\}$ is isomorphic to $\mathfrak{gl}(m|n)$.

Let $\tilde{\mathfrak{h}}$ stand for the Cartan subalgebra spanned by $\{E_{rr} | r \in \tilde{\mathbb{I}}\}$, and let $\tilde{\mathfrak{h}}^*$ denote its restricted dual. We may regard the elements $\epsilon_i^{\mathbf{b}}$ ($1 \leq i \leq m+n$) as elements in $\tilde{\mathfrak{h}}^*$ in a natural way. For $r \in \frac{1}{2}\mathbb{N}$, define $\delta_r \in \tilde{\mathfrak{h}}^*$ to be the element determined by

$$\delta_r(E_{ss}) = \delta_{rs}, \quad s \in \tilde{\mathbb{I}},$$

so that $\{\epsilon_i^{\mathbf{b}}, \delta_r | 1 \leq i \leq m+n, r \in \frac{1}{2}\mathbb{N}\}$ is a basis for $\tilde{\mathfrak{h}}^*$. Denote the set of roots of $\tilde{\mathfrak{g}}$ by $\tilde{\Phi}$. The ordered basis $\{\epsilon_1^{\mathbf{b}}, \dots, \epsilon_{m+n}^{\mathbf{b}}, e_{\frac{1}{2}}, e_1, \dots\}$ of \tilde{V} determines a Borel subalgebra $\tilde{\mathcal{B}}_{\mathbf{b}}$ with the simple system

$$\Pi(\tilde{\mathcal{B}}_{\mathbf{b}}) = \{\epsilon_1^{\mathbf{b}} - \epsilon_2^{\mathbf{b}}, \dots, \epsilon_{m+n-1}^{\mathbf{b}} - \epsilon_{m+n}^{\mathbf{b}}\} \cup \{\epsilon_{m+n}^{\mathbf{b}} - \delta_{\frac{1}{2}}\} \cup \{\delta_r - \delta_{r+\frac{1}{2}} | r \in \frac{1}{2}\mathbb{N}\}.$$

Denote the Dynkin diagram of the Lie superalgebra $\mathfrak{gl}(m|n)$ with respect to the Borel subalgebra \mathfrak{b} by $(\mathfrak{T}^{\mathbf{b}})$. Then the corresponding Dynkin diagram of $\tilde{\mathcal{B}}_{\mathbf{b}}$ together with $\Pi(\tilde{\mathcal{B}}_{\mathbf{b}})$ is given by

$$\begin{array}{ll} \begin{array}{c} \boxed{\mathfrak{T}^{\mathbf{b}}} \text{---} \bigotimes \text{---} \bigotimes \text{---} \bigotimes \text{---} \cdots \\ \epsilon_{m+1}^{\mathbf{b}} - \delta_{\frac{1}{2}} \quad \delta_{\frac{1}{2}} - \delta_1 \quad \delta_1 - \delta_{\frac{3}{2}} \end{array} & \text{if } b_{m+n} = 0; \\ \begin{array}{c} \boxed{\mathfrak{T}^{\mathbf{b}}} \text{---} \bigcirc \text{---} \bigotimes \text{---} \bigotimes \text{---} \cdots \\ \epsilon_{m+1}^{\mathbf{b}} - \delta_{\frac{1}{2}} \quad \delta_{\frac{1}{2}} - \delta_1 \quad \delta_1 - \delta_{\frac{3}{2}} \end{array} & \text{if } b_{m+n} = 1. \end{array}$$

Let \mathfrak{g} and $\check{\mathfrak{g}}$ be the Lie subalgebras of $\tilde{\mathfrak{g}}$ spanned by $\{E_{rs} | r, s \in \mathbb{I}\}$ and $\{E_{rs} | r, s \in \check{\mathbb{I}}\}$, respectively. The Cartan subalgebras of \mathfrak{g} and $\check{\mathfrak{g}}$ are $\mathfrak{h} = \tilde{\mathfrak{h}} \cap \mathfrak{g}$ and $\check{\mathfrak{h}} = \tilde{\mathfrak{h}} \cap \check{\mathfrak{g}}$, and their restricted duals are denoted by \mathfrak{h}^* and $\check{\mathfrak{h}}^*$, respectively. The subalgebras $\mathcal{B}_{\mathbf{b}} = \tilde{\mathcal{B}}_{\mathbf{b}} \cap \mathfrak{g}$ and $\check{\mathcal{B}}_{\mathbf{b}} = \tilde{\mathcal{B}}_{\mathbf{b}} \cap \check{\mathfrak{g}}$ are Borel subalgebras of \mathfrak{g} and $\check{\mathfrak{g}}$, respectively. The simple systems of \mathfrak{g} and $\check{\mathfrak{g}}$ with respect to $\mathcal{B}_{\mathbf{b}}$ and $\check{\mathcal{B}}_{\mathbf{b}}$ are denoted by $\Pi(\mathcal{B}_{\mathbf{b}})$ and $\Pi(\check{\mathcal{B}}_{\mathbf{b}})$, respectively. The Dynkin diagrams with $\Pi(\check{\mathcal{B}}_{\mathbf{b}})$ and $\Pi(\mathcal{B}_{\mathbf{b}})$ specified are as follows.

$$\begin{array}{ll} \mathfrak{g} : \begin{array}{c} \boxed{\mathfrak{T}^{\mathbf{b}}} \text{---} \bigcirc \text{---} \bigcirc \text{---} \bigcirc \text{---} \cdots \\ \epsilon_{m+1}^{\mathbf{b}} - \delta_1 \quad \delta_1 - \delta_2 \quad \delta_2 - \delta_3 \end{array} & \text{if } b_{m+n} = 0; \\ \check{\mathfrak{g}} : \begin{array}{c} \boxed{\mathfrak{T}^{\mathbf{b}}} \text{---} \bigotimes \text{---} \bigcirc \text{---} \bigcirc \text{---} \cdots \\ \epsilon_{m+1}^{\mathbf{b}} - \delta_{\frac{1}{2}} \quad \delta_{\frac{1}{2}} - \delta_{\frac{3}{2}} \quad \delta_{\frac{3}{2}} - \delta_{\frac{5}{2}} \end{array} & \text{if } b_{m+n} = 0; \\ \mathfrak{g} : \begin{array}{c} \boxed{\mathfrak{T}^{\mathbf{b}}} \text{---} \bigotimes \text{---} \bigcirc \text{---} \bigcirc \text{---} \cdots \\ \epsilon_{m+1}^{\mathbf{b}} - \delta_1 \quad \delta_1 - \delta_2 \quad \delta_2 - \delta_3 \end{array} & \text{if } b_{m+n} = 1; \\ \check{\mathfrak{g}} : \begin{array}{c} \boxed{\mathfrak{T}^{\mathbf{b}}} \text{---} \bigcirc \text{---} \bigcirc \text{---} \bigcirc \text{---} \cdots \\ \epsilon_{m+1}^{\mathbf{b}} - \delta_{\frac{1}{2}} \quad \delta_{\frac{1}{2}} - \delta_{\frac{3}{2}} \quad \delta_{\frac{3}{2}} - \delta_{\frac{5}{2}} \end{array} & \text{if } b_{m+n} = 1. \end{array}$$

7.2. Parabolic categories. We define

$$\begin{aligned}
 \widetilde{X} &:= \left\{ \sum_{i=1}^{m+n} \lambda_i \epsilon_i + \sum_{r \in \frac{1}{2}\mathbb{N}} {}^+ \lambda_r \delta_r \mid \lambda_i \in \mathbb{Z} \text{ and } {}^+ \lambda_r \in \mathbb{Z} \right\} \subseteq \widetilde{\mathfrak{h}}^*, \\
 (7.1) \quad X &:= \left\{ \sum_{i=1}^{m+n} \lambda_i \epsilon_i + \sum_{j \in \mathbb{N}} {}^+ \lambda_j \delta_j \mid \lambda_i \in \mathbb{Z} \text{ and } {}^+ \lambda_j \in \mathbb{Z} \right\} \subseteq \mathfrak{h}^*, \\
 \check{X} &:= \left\{ \sum_{i=1}^{m+n} \lambda_i \epsilon_i + \sum_{s \in \frac{1}{2} + \mathbb{Z}_+} {}^+ \lambda_s \delta_s \mid \lambda_i \in \mathbb{Z} \text{ and } {}^+ \lambda_s \in \mathbb{Z} \right\} \subseteq \check{\mathfrak{h}}^*.
 \end{aligned}$$

We shall identify $\lambda = \sum_{i=1}^{m+n} \lambda_i \epsilon_i + \sum_{j \in \mathbb{N}} {}^+ \lambda_j \delta_j \in X$ with the tuple $(\lambda_1, \dots, \lambda_{m+n}; {}^+ \lambda)$, where we write ${}^+ \lambda = ({}^+ \lambda_1, {}^+ \lambda_2, \dots)$. Recall \mathcal{P} denotes the set of all partitions. We let

$$(7.2) \quad X^+ := \left\{ \sum_{i=1}^{m+n} \lambda_i \epsilon_i + \sum_{j \in \mathbb{N}} {}^+ \lambda_j \delta_j \mid \lambda_i \in \mathbb{Z}, ({}^+ \lambda_1, {}^+ \lambda_2, \dots) \in \mathcal{P} \right\} \subseteq X.$$

For a partition $\mu = (\mu_1, \mu_2, \dots)$, let $\mu' = (\mu'_1, \mu'_2, \dots)$ denote the conjugate partition of μ . We also define $\theta(\mu)$ to be the sequence of integers (which is a variant of the Frobenius notation of μ)

$$\theta(\mu) := (\theta(\mu)_{1/2}, \theta(\mu)_1, \theta(\mu)_{3/2}, \theta(\mu)_2, \dots),$$

where

$$\theta(\mu)_{i-1/2} := \max\{\mu'_i - (i-1), 0\}, \quad \theta(\mu)_i := \max\{\mu_i - i, 0\}, \quad \forall i \in \mathbb{N}.$$

We identify elements in X^+ with tuples in $\mathbb{Z}^{m+n} \times \mathcal{P}$. For $\lambda = (\lambda_1, \dots, \lambda_{m+n}, {}^+ \lambda) \in X^+$, define

$$\begin{aligned}
 \lambda^\theta &:= \sum_{i=1}^{m+n} \lambda_i \epsilon_i + \sum_{r \in \frac{1}{2}\mathbb{N}} \theta({}^+ \lambda)_r \delta_r \in \widetilde{\mathfrak{h}}^*, \\
 (7.3) \quad \lambda^\natural &:= \sum_{i=1}^{m+n} \lambda_i \epsilon_i + \sum_{s \in \frac{1}{2} + \mathbb{Z}_+} ({}^+ \lambda)'_{s+\frac{1}{2}} \delta_s \in \check{\mathfrak{h}}^*.
 \end{aligned}$$

Furthermore we set

$$(7.4) \quad \widetilde{X}^+ := \{\lambda^\theta \mid \lambda \in X^+\}, \quad \check{X}^+ := \{\lambda^\natural \mid \lambda \in X^+\}$$

so that we have natural bijections

$$\check{X}^+ \xleftarrow{\natural} X^+ \xleftarrow{\theta} \widetilde{X}^+, \quad \lambda^\natural \leftrightarrow \lambda \leftrightarrow \lambda^\theta.$$

For a root α of $\tilde{\mathfrak{g}}$ we denote by $\tilde{\mathfrak{g}}_\alpha$ the root space corresponding to α . Similar notations apply to \mathfrak{g}_α and $\check{\mathfrak{g}}_\alpha$. Define

$$\begin{aligned}\tilde{\mathfrak{k}} &:= \tilde{\mathfrak{h}} + \sum_{\alpha \in \tilde{\Phi} \cap \sum_{r \in \tilde{\mathbb{I}}} \mathbb{Z}\delta_r} \tilde{\mathfrak{g}}_\alpha, & \tilde{\mathfrak{p}}_{\mathbf{b}} &:= \tilde{\mathcal{B}}_{\mathbf{b}} + \tilde{\mathfrak{k}}, \\ \check{\mathfrak{k}} &:= \check{\mathfrak{h}} + \sum_{\alpha \in \tilde{\Phi} \cap \sum_{r \in \tilde{\mathbb{I}}} \mathbb{Z}\delta_r} \check{\mathfrak{g}}_\alpha, & \check{\mathfrak{p}}_{\mathbf{b}} &:= \check{\mathcal{B}}_{\mathbf{b}} + \check{\mathfrak{k}}, \\ \mathfrak{k} &:= \mathfrak{h} + \sum_{\alpha \in \tilde{\Phi} \cap \sum_{r \in \tilde{\mathbb{I}}} \mathbb{Z}\delta_r} \mathfrak{g}_\alpha, & \mathfrak{p}_{\mathbf{b}} &:= \mathcal{B}_{\mathbf{b}} + \mathfrak{k}.\end{aligned}$$

Then $\tilde{\mathfrak{p}}_{\mathbf{b}}$, $\check{\mathfrak{p}}_{\mathbf{b}}$ and $\mathfrak{p}_{\mathbf{b}}$ are parabolic subalgebras of $\tilde{\mathfrak{g}}$, $\check{\mathfrak{g}}$ and \mathfrak{g} with Levi subalgebras $\tilde{\mathfrak{k}}$, $\check{\mathfrak{k}}$, and \mathfrak{k} , respectively. Let us denote the respective nilradicals and opposite nilradicals by $\tilde{\mathfrak{u}}_{\mathbf{b}}$, $\check{\mathfrak{u}}_{\mathbf{b}}$, and $\mathfrak{u}_{\mathbf{b}}$, and $\tilde{\mathfrak{u}}_{\mathbf{b}}^-$, $\check{\mathfrak{u}}_{\mathbf{b}}^-$, and $\mathfrak{u}_{\mathbf{b}}^-$.

For $\lambda \in X^+$, let $\tilde{L}^0(\lambda^\theta)$, $\check{L}^0(\lambda^\natural)$, and $L^0(\lambda)$ denote the irreducible $\tilde{\mathfrak{k}}$ -, $\check{\mathfrak{k}}$ -, and \mathfrak{k} -module of highest weight λ^θ , λ^\natural , and λ , respectively. They can be extended in a trivial way to $\tilde{\mathfrak{p}}_{\mathbf{b}}$ -, $\check{\mathfrak{p}}_{\mathbf{b}}$ -, and $\mathfrak{p}_{\mathbf{b}}$ -modules, respectively. We form the respective parabolic Verma modules

$$\tilde{\mathcal{M}}_{\mathbf{b}}(\lambda^\theta) = \text{Ind}_{\tilde{\mathfrak{p}}_{\mathbf{b}}}^{\tilde{\mathfrak{g}}} \tilde{L}^0(\lambda^\theta), \quad \check{\mathcal{M}}_{\mathbf{b}}(\lambda^\natural) = \text{Ind}_{\check{\mathfrak{p}}_{\mathbf{b}}}^{\check{\mathfrak{g}}} \check{L}^0(\lambda^\natural), \quad \mathcal{M}_{\mathbf{b}}(\lambda) = \text{Ind}_{\mathfrak{p}_{\mathbf{b}}}^{\mathfrak{g}} L^0(\lambda),$$

whose unique irreducible quotients are denoted by $\tilde{\mathcal{L}}_{\mathbf{b}}(\lambda^\theta)$, $\check{\mathcal{L}}_{\mathbf{b}}(\lambda^\natural)$, and $\mathcal{L}_{\mathbf{b}}(\lambda)$, respectively.

Definition 7.1. Let $\tilde{\mathcal{O}}_{\mathbf{b}}$ be the category of finitely generated $\tilde{\mathfrak{g}}$ -modules $\tilde{\mathcal{M}}$ such that $\tilde{\mathcal{M}}$ is a semisimple $\tilde{\mathfrak{h}}$ -module with finite-dimensional weight subspaces $\tilde{\mathcal{M}}_\gamma$, $\gamma \in \tilde{X}$, satisfying the following conditions.

- (i) $\tilde{\mathcal{M}}$ decomposes over $\tilde{\mathfrak{k}}$ into a direct sum of $\tilde{L}^0(\mu^\theta)$ for $\mu \in X^+$.
- (ii) There exist finitely many weights ${}^1\lambda, {}^2\lambda, \dots, {}^t\lambda \in X^+$ (depending on $\tilde{\mathcal{M}}$) such that if γ is a weight in $\tilde{\mathcal{M}}$, then $\gamma \in {}^i\lambda^\theta - \sum_{\alpha \in \Pi(\tilde{\mathcal{B}}_{\mathbf{b}})} \mathbb{Z}_+\alpha$, for some i .

The morphisms in $\tilde{\mathcal{O}}_{\mathbf{b}}$ are all (not necessarily even) homomorphisms of $\tilde{\mathfrak{g}}$ -modules.

Let $\mathcal{M} \in \tilde{\mathcal{O}}_{\mathbf{b}}$ so that $\mathcal{M} = \bigoplus_{\gamma \in \tilde{X}} \mathcal{M}_\gamma$. For $\vartheta \in \mathbb{Z}_2$ let

$$\tilde{X}_\vartheta = \left\{ \gamma \in \tilde{X} \mid \sum_{i=m+1}^n \langle \gamma, e_{ii} \rangle + \sum_{r \in \frac{1}{2} + \mathbb{Z}_+} \langle \gamma, E_{rr} \rangle \equiv \vartheta \right\}.$$

We define as in §6.1

$$(7.5) \quad \mathcal{M}' = \mathcal{M}'_0 \oplus \mathcal{M}'_1, \quad \text{where } \mathcal{M}'_\vartheta := \bigoplus_{\gamma \in \tilde{X}_\vartheta} \mathcal{M}_\gamma \quad (\vartheta \in \mathbb{Z}_2).$$

Then $\mathcal{M}' \in \tilde{\mathcal{O}}_{\mathbf{b}}$, and by definition $\mathcal{M} \cong \mathcal{M}'$. As argued in §6.1, the full subcategory $\tilde{\mathcal{O}}_{\mathbf{b}, \bar{0}}$ of $\tilde{\mathcal{O}}_{\mathbf{b}}$ consisting of objects with \mathbb{Z}_2 -gradation given by (7.5) is an abelian category. Since the categories $\tilde{\mathcal{O}}_{\mathbf{b}, \bar{0}}$ and $\tilde{\mathcal{O}}_{\mathbf{b}}$ have isomorphic skeleton categories, we conclude that $\tilde{\mathcal{O}}_{\mathbf{b}}$ is an abelian category.

The abelian categories $\check{\mathcal{O}}_{\mathbf{b}}$ of $\check{\mathfrak{g}}$ -modules and $\mathcal{O}_{\mathbf{b}}$ of \mathfrak{g} -modules are defined in a similar fashion.

The modules $\tilde{\mathcal{M}}_{\mathbf{b}}(\lambda^\theta)$ and $\tilde{\mathcal{L}}_{\mathbf{b}}(\lambda^\theta)$, $\check{\mathcal{M}}_{\mathbf{b}}(\lambda^\natural)$ and $\check{\mathcal{L}}_{\mathbf{b}}(\lambda^\natural)$, $\mathcal{M}_{\mathbf{b}}(\lambda)$ and $\mathcal{L}_{\mathbf{b}}(\lambda)$ lie in the categories $\tilde{\mathcal{O}}_{\mathbf{b}}$, $\check{\mathcal{O}}_{\mathbf{b}}$, $\mathcal{O}_{\mathbf{b}}$, respectively, for $\lambda \in X^+$. As in Definition 6.6 we can also define, for each $\lambda \in X^+$, tilting modules $\tilde{\mathcal{T}}_{\mathbf{b}}(\lambda^\theta)$, $\check{\mathcal{T}}_{\mathbf{b}}(\lambda^\natural)$, and $\mathcal{T}_{\mathbf{b}}(\lambda)$ in the categories $\tilde{\mathcal{O}}_{\mathbf{b}}$, $\check{\mathcal{O}}_{\mathbf{b}}$, and $\mathcal{O}_{\mathbf{b}}$, respectively. We can now adapt the arguments in [CW1] and [?, Section 5] to show that tilting modules exist and are unique in these respective categories. In contrast to the more standard setups in [So2, Br2], the Lie superalgebras under considerations here are infinite-rank.

For $\tilde{\mathcal{M}} \in \tilde{\mathcal{O}}_{\mathbf{b}}$ we denote the n th $\tilde{\mathfrak{u}}_{\mathbf{b}}$ -homology group with coefficients in $\tilde{\mathcal{M}}$ by $H_n(\tilde{\mathfrak{u}}_{\mathbf{b}}^-; \tilde{\mathcal{M}})$. For $\check{\mathcal{M}} \in \check{\mathcal{O}}_{\mathbf{b}}$ and $\mathcal{M} \in \mathcal{O}_{\mathbf{b}}$ the notations $H_n(\check{\mathfrak{u}}_{\mathbf{b}}^-; \check{\mathcal{M}})$ and $H_n(\mathfrak{u}_{\mathbf{b}}^-; \mathcal{M})$ stand for similar homology groups. We introduce the following, for $\lambda, \mu \in X^+$:

$$(7.6) \quad \begin{aligned} \tilde{l}_{\mu^\theta \lambda^\theta}^{\mathbf{b}}(q) &:= \sum_{i=0}^{\infty} \dim \operatorname{Hom}_{\check{\mathfrak{k}}} \left(\tilde{L}^0(\mu^\theta), H_i \left(\tilde{\mathfrak{u}}_{\mathbf{b}}^-; \tilde{\mathcal{L}}_{\mathbf{b}}(\lambda^\theta) \right) \right) (-q^{-1})^i, \\ \check{l}_{\mu^\natural \lambda^\natural}^{\mathbf{b}}(q) &:= \sum_{i=0}^{\infty} \dim \operatorname{Hom}_{\check{\mathfrak{k}}} \left(\check{L}^0(\mu^\natural), H_i \left(\check{\mathfrak{u}}_{\mathbf{b}}^-; \check{\mathcal{L}}_{\mathbf{b}}(\lambda^\natural) \right) \right) (-q^{-1})^i, \\ l_{\mu \lambda}^{\mathbf{b}}(q) &:= \sum_{i=0}^{\infty} \dim \operatorname{Hom}_{\mathfrak{k}} \left(L^0(\mu), H_i \left(\mathfrak{u}_{\mathbf{b}}^-; \mathcal{L}_{\mathbf{b}}(\lambda) \right) \right) (-q^{-1})^i. \end{aligned}$$

They turn out to be polynomials, and will be called *Kazhdan-Lusztig-Vogan (KLV) polynomials*.

7.3. Equivalence of categories. We may regard $X \subseteq \tilde{X}$ and $\check{X} \subseteq \tilde{X}$, by definitions given in (7.1). Given a semisimple $\tilde{\mathfrak{h}}$ -module $\tilde{\mathcal{M}}$ such that $\tilde{\mathcal{M}} = \bigoplus_{\gamma \in \tilde{X}} \tilde{\mathcal{M}}_\gamma$, we define

$$T(\tilde{\mathcal{M}}) := \bigoplus_{\gamma \in X} \tilde{\mathcal{M}}_\gamma, \quad \text{and} \quad \check{T}(\tilde{\mathcal{M}}) := \bigoplus_{\gamma \in \check{X}} \tilde{\mathcal{M}}_\gamma.$$

Note that $T(\tilde{\mathcal{M}})$ is an \mathfrak{h} -submodule of $\tilde{\mathcal{M}}$ (regarded as an \mathfrak{h} -module), and $\check{T}(\tilde{\mathcal{M}})$ is an $\check{\mathfrak{h}}$ -submodule of $\tilde{\mathcal{M}}$ (regarded as an $\check{\mathfrak{h}}$ -module). Also if $\tilde{\mathcal{M}}$ is a $\tilde{\mathfrak{k}}$ -module, then $T(\tilde{\mathcal{M}})$ is a \mathfrak{k} -submodule of $\tilde{\mathcal{M}}$ (regarded as a \mathfrak{k} -module), and $\check{T}(\tilde{\mathcal{M}})$ is a $\check{\mathfrak{k}}$ -submodule of $\tilde{\mathcal{M}}$ (regarded as a $\check{\mathfrak{k}}$ -module). Furthermore if $\tilde{\mathcal{M}} \in \tilde{\mathcal{O}}_{\mathbf{b}}$, then $T(\tilde{\mathcal{M}})$ is a \mathfrak{g} -submodule of $\tilde{\mathcal{M}}$ (regarded as a \mathfrak{g} -module), and $\check{T}(\tilde{\mathcal{M}})$ is a $\check{\mathfrak{g}}$ -submodule of $\tilde{\mathcal{M}}$ (regarded as a $\check{\mathfrak{g}}$ -module).

If $\tilde{f} : \tilde{\mathcal{M}} \rightarrow \tilde{\mathcal{N}}$ is an $\tilde{\mathfrak{h}}$ -homomorphism, we let

$$T[\tilde{f}] : T(\tilde{\mathcal{M}}) \longrightarrow T(\tilde{\mathcal{N}}) \quad \text{and} \quad \check{T}[\tilde{f}] : \check{T}(\tilde{\mathcal{M}}) \longrightarrow \check{T}(\tilde{\mathcal{N}})$$

be the corresponding restriction maps. Then $T[\tilde{f}]$ (respectively, $\check{T}[\tilde{f}]$) is an \mathfrak{h} - (respectively, $\check{\mathfrak{h}}$ -) homomorphism. If \tilde{f} is also a homomorphism of $\tilde{\mathfrak{k}}$ -modules, then $T[\tilde{f}]$ (respectively, $\check{T}[\tilde{f}]$) is a \mathfrak{k} - (respectively, $\check{\mathfrak{k}}$ -) homomorphism. Finally, if \tilde{f} is also a homomorphism of $\tilde{\mathfrak{g}}$ -modules, then $T[\tilde{f}]$ (respectively, $\check{T}[\tilde{f}]$) is a \mathfrak{g} - (respectively, $\check{\mathfrak{g}}$ -) homomorphism.

Recall the notations λ^θ , λ^\natural from (7.3). Following the line of arguments of [CL, ?] we can show that T and \check{T} define exact functors from $\tilde{\mathcal{O}}_{\mathbf{b}}$ to $\mathcal{O}_{\mathbf{b}}$ and from $\tilde{\mathcal{O}}_{\mathbf{b}}$ to $\check{\mathcal{O}}_{\mathbf{b}}$, respectively; moreover, we establish the following theorem similarly.

Theorem 7.2 (Super Duality). (1) $T : \tilde{\mathcal{O}}_{\mathbf{b}} \rightarrow \mathcal{O}_{\mathbf{b}}$ and $\check{T} : \tilde{\mathcal{O}}_{\mathbf{b}} \rightarrow \check{\mathcal{O}}_{\mathbf{b}}$ are equivalences of categories. Consequently, the categories $\mathcal{O}_{\mathbf{b}}$ and $\check{\mathcal{O}}_{\mathbf{b}}$ are equivalent.

(2) For $\mathcal{Y} = \mathcal{M}, \mathcal{L}, \mathcal{T}$ and $\lambda \in X^+$, we have $T(\tilde{\mathcal{Y}}_{\mathbf{b}}(\lambda^\theta)) = \mathcal{Y}_{\mathbf{b}}(\lambda)$, $\check{T}(\tilde{\mathcal{Y}}_{\mathbf{b}}(\lambda^\theta)) = \check{\mathcal{Y}}_{\mathbf{b}}(\lambda^\natural)$. Consequently, for $\mathcal{X} = \mathcal{L}, \mathcal{T}$, we have

$$\begin{aligned} \text{ch}\mathcal{X}_{\mathbf{b}}(\lambda) &= \sum_{\mu \in X^+} a_{\mu\lambda} \text{ch}\mathcal{M}_{\mathbf{b}}(\mu) \iff \text{ch}\tilde{\mathcal{X}}_{\mathbf{b}}(\lambda^\theta) = \sum_{\mu \in X^+} a_{\mu\lambda} \text{ch}\tilde{\mathcal{M}}_{\mathbf{b}}(\mu^\theta) \\ &\iff \text{ch}\check{\mathcal{X}}_{\mathbf{b}}(\lambda^\natural) = \sum_{\mu \in X^+} a_{\mu\lambda} \text{ch}\check{\mathcal{M}}_{\mathbf{b}}(\mu^\natural), \quad \text{for } a_{\mu\lambda} \in \mathbb{Z}. \end{aligned}$$

(3) For $\lambda, \mu \in X^+$ the functors T and \check{T} induces natural isomorphisms

$$\begin{aligned} \text{Hom}_{\check{\mathfrak{k}}}(\check{L}^0(\mu^\theta), H_n(\check{\mathfrak{u}}_{\mathbf{b}}^-, \check{\mathcal{L}}_{\mathbf{b}}(\lambda^\theta))) &\cong \text{Hom}_{\check{\mathfrak{k}}}(\check{L}^0(\mu^\natural), H_n(\check{\mathfrak{u}}_{\mathbf{b}}^-, \check{\mathcal{L}}_{\mathbf{b}}(\lambda^\natural))) \\ &\cong \text{Hom}_{\mathfrak{k}}(L^0(\mu), H_n(\mathfrak{u}_{\mathbf{b}}^-, \mathcal{L}_{\mathbf{b}}(\lambda))). \end{aligned}$$

Consequently, the corresponding Kazhdan-Lusztig-Vogan polynomials are identical, that is, $\check{\mathfrak{l}}_{\mu^\theta\lambda^\theta}^{\mathbf{b}}(q) = \check{\mathfrak{l}}_{\mu^\natural\lambda^\natural}^{\mathbf{b}}(q) = \mathfrak{l}_{\mu\lambda}^{\mathbf{b}}(q)$.

Remark 7.3. Theorem 7.2 affords further parabolic variants which allow more general even Levi subalgebras on the $(\mathfrak{Z}^{\mathbf{b}})$ side (see the Dynkin diagrams in §7.1). A novel viewpoint of Theorem 7.2, in contrast to [CL, CLW], is that super duality holds also when the *head* Dynkin diagram is that of a Lie *superalgebra*.

Remark 7.4. Using the same argument as [CL, Proposition 3.11], we can show that $\check{\mathcal{M}}_{\mathbf{b}}(\lambda^\natural)$, for $\lambda \in X^+$, has a finite composition series with composition factors with highest weights lying in \check{X}^+ . It follows therefore that $\check{\mathcal{O}}_{\mathbf{b}}$ is the category of $\check{\mathfrak{g}}$ -modules that have finite composition series and that, as $\check{\mathfrak{k}}$ -modules, decompose into direct sums of irreducible $\check{\mathfrak{k}}$ -modules with highest weights lying in \check{X}^+ . Thus, the categories $\check{\mathcal{O}}_{\mathbf{b}}$ are independent of the choices of the $0^m 1^n$ -sequences \mathbf{b} . Similarly the categories $\tilde{\mathcal{O}}_{\mathbf{b}}$ (and $\mathcal{O}_{\mathbf{b}}$, respectively) are all independent of the choices of \mathbf{b} .

7.4. BGG categories of finite rank. For $k \in \mathbb{N}$, we let \mathfrak{g}^k and $\check{\mathfrak{g}}^k$ be the respective finite-dimensional general linear Lie superalgebras with simple roots as follows:

$$\begin{aligned} \{\epsilon_1^{\mathbf{b}} - \epsilon_2^{\mathbf{b}}, \dots, \epsilon_{m+n-1}^{\mathbf{b}} - \epsilon_{m+n}^{\mathbf{b}}, \epsilon_{m+1}^{\mathbf{b}} - \delta_1, \delta_1 - \delta_2, \dots, \delta_{k-1} - \delta_k\}, \\ \{\epsilon_1^{\mathbf{b}} - \epsilon_2^{\mathbf{b}}, \dots, \epsilon_{m+n-1}^{\mathbf{b}} - \epsilon_{m+n}^{\mathbf{b}}, \epsilon_{m+1}^{\mathbf{b}} - \delta_{1/2}, \delta_{1/2} - \delta_{3/2}, \dots, \delta_{k-3/2} - \delta_{k-1/2}\}. \end{aligned}$$

Then \mathfrak{g}^k and $\check{\mathfrak{g}}^k$ may be regarded as subalgebras of \mathfrak{g} and $\check{\mathfrak{g}}$. We denote the standard Borel subalgebras corresponding to these simple systems by $\mathcal{B}_{\mathbf{b}}^k$ and $\check{\mathcal{B}}_{\mathbf{b}}^k$, respectively, and furthermore set $\mathfrak{h}^k = \mathfrak{h} \cap \mathfrak{g}^k$ and $\check{\mathfrak{h}}^k = \check{\mathfrak{h}} \cap \check{\mathfrak{g}}^k$. Moreover, we have natural inclusions $\mathfrak{g}^k \subseteq \mathfrak{g}^{k+1}$ and $\check{\mathfrak{g}}^k \subseteq \check{\mathfrak{g}}^{k+1}$, with $\mathfrak{g} = \bigcup_k \mathfrak{g}^k$ and $\check{\mathfrak{g}} = \bigcup_k \check{\mathfrak{g}}^k$.

Set

$$(7.7) \quad X^k = X \cap (\mathfrak{h}^k)^*, \quad \check{X}^k = \check{X} \cap (\check{\mathfrak{h}}^k)^*.$$

Also define

$$(7.8) \quad \begin{aligned} X^{k,+} &= \left\{ \lambda = \sum_{j=1}^{m+n} \lambda_j \epsilon_j^{\mathbf{b}} + \sum_{i=1}^k \mu_i \delta_i \in X^k \mid \mu_1 \geq \dots \geq \mu_k \right\}, \\ \check{X}^{k,+} &= \left\{ \sum_{j=1}^{m+n} \lambda_j \epsilon_j^{\mathbf{b}} + \sum_{i=1}^k \nu_i \delta_{i-\frac{1}{2}} \in \check{X}^k \mid \nu_1 \geq \dots \geq \nu_k \right\}. \end{aligned}$$

We shall identify a weight $\lambda \in X^{k,+}$ as the tuple $\lambda = (\lambda_1, \dots, \lambda_{m+n}; \mu_1, \dots, \mu_k)$. Given $\lambda = (\lambda_1, \dots, \lambda_{m+n}; {}^+\lambda) \in X^+$ with ${}^+\lambda_j = 0$ for $j > k$, we may regard λ as a weight in $X^{k,+}$ in a natural way. Similarly, for $\lambda \in X^+$ with ${}^+\lambda'_j = 0$ for $j > k$, we regard λ^{\natural} as a weight in $\check{X}^{k,+}$.

The Levi subalgebra, parabolic subalgebra, and the nilradical of the finite-rank Lie superalgebra \mathfrak{g}^k are

$$\mathfrak{k}^k = \mathfrak{k} \cap \mathfrak{g}^k, \quad \mathfrak{p}_{\mathbf{b}}^k = \mathfrak{p}_{\mathbf{b}} \cap \mathfrak{g}^k, \quad \mathfrak{u}_{\mathbf{b}}^k = \mathfrak{u}_{\mathbf{b}} \cap \mathfrak{g}^k,$$

respectively. We denote the parabolic Verma, irreducible, and tilting \mathfrak{g}^k -modules by $\mathcal{M}_{\mathbf{b}}^k(\lambda)$, $\mathcal{L}_{\mathbf{b}}^k(\lambda)$, and $\mathcal{T}_{\mathbf{b}}^k(\lambda)$, for $\lambda \in X^{k,+}$. The corresponding parabolic BGG category of \mathfrak{g}^k -modules is denoted by $\mathcal{O}_{\mathbf{b}}^k$, which is defined similarly as in Definition 7.1, now with \mathfrak{h} , \mathfrak{k} , et cetera therein replaced by \mathfrak{h}^k , \mathfrak{k}^k , et cetera.

The statements in the previous paragraph have obvious counterparts for the Lie superalgebra $\check{\mathfrak{g}}^k$ as well. We introduce the self-explanatory notations $\check{\mathcal{M}}_{\mathbf{b}}^k(\xi)$, $\check{\mathcal{L}}_{\mathbf{b}}^k(\xi)$, $\check{\mathcal{T}}_{\mathbf{b}}^k(\xi)$, $\check{\mathcal{O}}_{\mathbf{b}}^k$, $\check{\mathfrak{k}}^k$, $\check{\mathfrak{p}}_{\mathbf{b}}^k$, where $\xi \in \check{X}^{k,+}$.

In an entirely analogous manner as in the definitions of $\mathfrak{l}_{\mu\lambda}^{\mathbf{b}}(q)$ and $\check{\mathfrak{l}}_{\mu^{\natural}\lambda^{\natural}}^{\mathbf{b}}(q)$ in (7.6), we can define the *Kazhdan-Lusztig-Vogan (KLV) polynomials* $\mathfrak{l}_{\mu\lambda}^{\mathbf{b},0}(q)$, for $\lambda, \mu \in X^{k,+}$, and $\check{\mathfrak{l}}_{\xi\eta}^{\mathbf{b},1}(q)$, for $\xi, \eta \in \check{X}^{k,+}$, in the categories $\mathcal{O}_{\mathbf{b}}^k$ and $\check{\mathcal{O}}_{\mathbf{b}}^k$, respectively.

7.5. Truncation functors. Let $\lambda \in X^+$, and write $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{m+n}; {}^+\lambda)$, where ${}^+\lambda \in \mathcal{P}$. Recall the parabolic Verma \mathfrak{g} -modules $\mathcal{M}_{\mathbf{b}}(\lambda)$, $\check{\mathcal{M}}_{\mathbf{b}}(\lambda)$, the irreducible modules $\mathcal{L}_{\mathbf{b}}(\lambda)$, $\check{\mathcal{L}}_{\mathbf{b}}(\lambda)$, and the tilting modules $\mathcal{T}_{\mathbf{b}}(\lambda)$, $\check{\mathcal{T}}_{\mathbf{b}}(\lambda)$ in the categories $\mathcal{O}_{\mathbf{b}}$ and $\check{\mathcal{O}}_{\mathbf{b}}$, respectively. Let $\mathcal{M} \in \mathcal{O}_{\mathbf{b}}$ and $\check{\mathcal{M}} \in \check{\mathcal{O}}_{\mathbf{b}}$. Then we have the weight space decompositions

$$\mathcal{M} = \bigoplus_{\mu \in X} \mathcal{M}_{\mu}, \quad \check{\mathcal{M}} = \bigoplus_{\mu \in \check{X}} \check{\mathcal{M}}_{\mu}.$$

We define an exact functor $\mathfrak{tr} : \mathcal{O}_{\mathbf{b}} \rightarrow \mathcal{O}_{\mathbf{b}}^k$ by

$$\mathfrak{tr}(\mathcal{M}) := \bigoplus_{\mu} \{ \mathcal{M}_{\mu} \mid (\mu, \delta_j) = 0, \forall j \geq k+1 \text{ and } j \in \mathbb{N} \}.$$

Similarly, we define an exact functor $\check{\mathfrak{tr}} : \check{\mathcal{O}}_{\mathbf{b}} \rightarrow \check{\mathcal{O}}_{\mathbf{b}}^k$ by

$$\check{\mathfrak{tr}}(\check{\mathcal{M}}) := \bigoplus_{\mu} \{ \check{\mathcal{M}}_{\mu} \mid (\mu, \delta_r) = 0, \forall r > k \text{ and } r \in \frac{1}{2} + \mathbb{Z}_+ \}.$$

We have the following.

Proposition 7.5. *The functors $\mathbf{tr} : \mathcal{O}_{\mathbf{b}} \rightarrow \mathcal{O}_{\mathbf{b}}^k$ and $\check{\mathbf{tr}} : \check{\mathcal{O}}_{\mathbf{b}} \rightarrow \check{\mathcal{O}}_{\mathbf{b}}^k$ satisfy the following: for $\mathcal{Y} = \mathcal{M}, \mathcal{L}, \mathcal{T}$ and $\lambda = (\lambda_1, \dots, \lambda_{m+n}; {}^+\lambda) \in X^+$,*

$$\begin{aligned} \mathbf{tr}(\mathcal{Y}_{\mathbf{b}}(\lambda)) &= \begin{cases} \mathcal{Y}_{\mathbf{b}}^k(\lambda), & \text{if } \ell({}^+\lambda) \leq k, \\ 0, & \text{otherwise.} \end{cases} \\ \check{\mathbf{tr}}(\check{\mathcal{Y}}_{\mathbf{b}}(\lambda^{\natural})) &= \begin{cases} \check{\mathcal{Y}}_{\mathbf{b}}^k(\lambda^{\natural}), & \text{if } \ell({}^+\lambda') \leq k, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Moreover, we have $\mathfrak{l}_{\mu\lambda}^{\mathbf{b}}(q) = \mathfrak{l}_{\mu\lambda}^{\mathbf{b},0}(q)$ for $\ell({}^+\lambda) \leq k$ and $\ell({}^+\mu) \leq k$, and $\check{\mathfrak{l}}_{\mu^{\natural}\lambda^{\natural}}^{\mathbf{b}}(q) = \check{\mathfrak{l}}_{\mu^{\natural}\lambda^{\natural}}^{\mathbf{b},1}(q)$ for $\ell({}^+\lambda') \leq k$ and $\ell({}^+\mu') \leq k$.

Proof. For $\mathcal{Y} = \mathcal{M}$, this is easy. For $\mathcal{Y} = \mathcal{L}$ the argument in [CWZ, Lemma 3.5, Corollary 3.6], or, with greater details in [CW2, Proposition 6.7], can be adapted easily to our settings here. For $\mathcal{Y} = \mathcal{T}$, it follows by the same type of arguments as in [CW1, Proposition 3.12].

The coincidence of KLV polynomials under the truncation functors is an immediate consequence of the property that the truncation functors commute with the differentials of the complexes for the respective homology groups (cf. [CW2, Theorem 6.31]). \square

Remark 7.6. Due to the conditions in the definitions (7.2) of X^+ and (7.8) of $X^{k,+}$, the \mathfrak{g}^k -modules $\mathcal{Y}_{\mathbf{b}}^k(\lambda)$, for $\mathcal{Y} = \mathcal{M}, \mathcal{L}, \mathcal{T}$, appear as images of \mathbf{tr} if and only if $\lambda = (\lambda_1, \dots, \lambda_{m+n}; \mu_1, \dots, \mu_k) \in X^{k,+}$ satisfies the additional condition that $\mu_k \geq 0$. Similar remarks apply to the images of $\check{\mathbf{tr}}$.

8. PROOF OF BRUNDAN-KAZHDAN-LUSZTIG CONJECTURE

In this section, we prove the Brundan-Kazhdan-Lusztig (BKL) conjecture for the BGG category \mathcal{O} of $\mathfrak{gl}(m|n)$ -modules which is formulated in terms of canonical and dual canonical bases on a Fock space $\mathbb{T}^{\mathbf{b}}$, associated with a $0^m 1^n$ -sequence \mathbf{b} . Our proof is built on a Fock space reformulation of classical Kazhdan-Lusztig theory for type A Lie algebras, the super duality, and a comparison of BKL conjecture for adjacent Borel subalgebras.

8.1. BKL conjecture. Let \mathbf{b} be a $0^m 1^n$ -sequence. Recall from Section 6 that the BGG category $\mathcal{O}^{m|n} = \mathcal{O}_{\mathbf{b}}^{m|n}$ of $\mathfrak{gl}(m|n)$ -modules contains the \mathbf{b} -Verma module $M_{\mathbf{b}}(\lambda)$, the \mathbf{b} -highest weight irreducible modules $L_{\mathbf{b}}(\lambda)$, and the \mathbf{b} -tilting modules $T_{\mathbf{b}}(\lambda)$ for $\lambda \in X(m|n)$. Recall also that $\mathcal{O}_{\mathbf{b}}^{m|n,\Delta}$ denotes the full subcategory of $\mathcal{O}_{\mathbf{b}}^{m|n}$ consisting of objects that have finite \mathbf{b} -Verma flags, and let $[\mathcal{O}_{\mathbf{b}}^{m|n,\Delta}]$ denote its Grothendieck group.

Recall furthermore the Fock space $\mathbb{T}^{\mathbf{b}}$ and its B -completion $\hat{\mathbb{T}}^{\mathbf{b}}$ with respect to the Bruhat ordering $\preceq_{\mathbf{b}}$ from Definition 3.2. Starting with a $\mathbb{Z}[q, q^{-1}]$ -lattice spanned by the standard monomial basis for the $\mathbb{Q}(q)$ -vector space $\mathbb{T}^{\mathbf{b}}$, we define by a base change to \mathbb{Z} the specialization at $q = 1$ of $\mathbb{T}^{\mathbf{b}}$, denoted by $\mathbb{T}_{\mathbb{Z}}^{\mathbf{b}}$. The B -completion $\hat{\mathbb{T}}_{\mathbb{Z}}^{\mathbf{b}}$ is defined as usual. For a standard, canonical, or dual canonical basis element u in $\mathbb{T}^{\mathbf{b}} \hat{\otimes} \wedge^{\infty} \mathbb{V}$, we shall denote by $u(1)$ the corresponding element in the specialization at $q = 1$. Similar

remarks on specialization at $q = 1$ and similar notations apply below to other variants of Fock spaces.

Recall the bijection $X(m|n) \rightarrow \mathbb{Z}^{m+n}$ given by $\lambda \mapsto f_\lambda^{\mathbf{b}}$ from (6.6). We have a natural \mathbb{Z} -linear isomorphism $\psi_{\mathbf{b}} : [\mathcal{O}_{\mathbf{b}}^{m|n, \Delta}] \rightarrow \mathbb{T}_{\mathbb{Z}}^{\mathbf{b}}$ given by $[M_{\mathbf{b}}(\lambda)] \mapsto M_{f_\lambda^{\mathbf{b}}}^{\mathbf{b}}(1)$. We define a completion $[[\mathcal{O}_{\mathbf{b}}^{m|n, \Delta}]]$ so that $\psi_{\mathbf{b}}$ extends to a \mathbb{Z} -linear isomorphism between the two completions

$$\psi_{\mathbf{b}} : [[\mathcal{O}_{\mathbf{b}}^{m|n, \Delta}]] \rightarrow \widehat{\mathbb{T}}_{\mathbb{Z}}^{\mathbf{b}}, \quad [M_{\mathbf{b}}(\lambda)] \mapsto M_{f_\lambda^{\mathbf{b}}}^{\mathbf{b}}(1).$$

We note that $[L_{\mathbf{b}}(\lambda)] \in [[\mathcal{O}_{\mathbf{b}}^{m|n, \Delta}]]$, though $L_{\mathbf{b}}(\lambda) \notin \mathcal{O}_{\mathbf{b}}^{m|n, \Delta}$ in general.

We now formulate Brundan's Kazhdan-Lusztig type conjecture for $\mathcal{O}_{\mathbf{b}}^{m|n}$, for an arbitrary $0^m 1^n$ -sequence \mathbf{b} . Recall the BKL polynomials $\ell_{f_\mu^{\mathbf{b}} f_\lambda^{\mathbf{b}}}(q)$ and $t_{f_\mu^{\mathbf{b}} f_\lambda^{\mathbf{b}}}(q)$ from Proposition 3.9.

Conjecture 8.1 (BKL conjecture). *Let \mathbf{b} be an arbitrary $0^m 1^n$ -sequence.*

(1) *We have $\psi_{\mathbf{b}}([L_{\mathbf{b}}(\lambda)]) = L_{f_\lambda^{\mathbf{b}}}^{\mathbf{b}}(1)$, for all $\lambda \in X(m|n)$. Equivalently, we have*

$$[L_{\mathbf{b}}(\lambda)] = \sum_{\mu} \ell_{f_\mu^{\mathbf{b}} f_\lambda^{\mathbf{b}}}(1) [M_{\mathbf{b}}(\mu)].$$

(2) *We have $\psi_{\mathbf{b}}([T_{\mathbf{b}}(\lambda)]) = T_{f_\lambda^{\mathbf{b}}}^{\mathbf{b}}(1)$, for all $\lambda \in X(m|n)$. Equivalently, we have*

$$[T_{\mathbf{b}}(\lambda)] = \sum_{\mu} t_{f_\mu^{\mathbf{b}} f_\lambda^{\mathbf{b}}}(1) [M_{\mathbf{b}}(\mu)].$$

Remark 8.2. Conjecture 8.1 for the standard $0^m 1^n$ -sequence $\mathbf{b}_{\text{st}} = (0^m, 1^n)$ is precisely [Br1, Conjecture 4.32], and the variants of Brundan's conjecture for general \mathbf{b} have been expected (cf. Kujawa's thesis [Ku]), though the completions of various Fock spaces and their canonical bases were not formulated in *loc. cit.*. Kujawa provided supporting evidence for the BKL conjecture by showing the irreducible modules in $\mathcal{O}^{m|n}$ form a crystal basis compatible with the one coming from $\mathbb{T}^{\mathbf{b}}$. When $n = 0$ or $m = 0$, the BKL conjecture reduces to a reformulation of classical Kazhdan-Lusztig conjecture (see Theorem 8.3 for $k = 0$ below).

The remainder of this paper is devoted to a proof of this conjecture. We will follow closely the strategy of proof outlined in §1.5.

8.2. Bijections. Let $k \in \mathbb{N} \cup \{\infty\}$. Recall $f_\lambda^{\mathbf{b}} \in \mathbb{Z}^{m+n}$ from (6.7), respectively. Also recall \mathbb{Z}_+^k and \mathbb{Z}_-^k from (2.12) and (2.14). The following maps

$$(8.1) \quad \begin{aligned} X^{k,+} &\longrightarrow \mathbb{Z}^{m+n} \times \mathbb{Z}_+^k, & \lambda &\mapsto f_\lambda^{\mathbf{b}0}, \\ \check{X}^{k,+} &\longrightarrow \mathbb{Z}^{m+n} \times \mathbb{Z}_-^k, & \lambda &\mapsto f_\lambda^{\mathbf{b}1}, \end{aligned}$$

are bijections, where $X^{\infty,+}$ and $\check{X}^{\infty,+}$ are understood to be X^+ and \check{X}^+ in (7.2) and (7.4), respectively. Here $f_\lambda^{\mathbf{b}0} \in \mathbb{Z}^{m+n} \times \mathbb{Z}_+^k$ and $f_\lambda^{\mathbf{b}1} \in \mathbb{Z}^{m+n} \times \mathbb{Z}_-^k$, for $\lambda =$

$(\lambda_1, \dots, \lambda_{m+n}; {}^+\lambda)$ with ${}^+\lambda = ({}^+\lambda_1, \dots, {}^+\lambda_k)$, are defined by setting

$$\begin{aligned} f_\lambda^{\mathbf{b}0}(i) &= f_\lambda^{\mathbf{b}}(i), & \text{if } i \in [m+n], \\ f_\lambda^{\mathbf{b}0}(\underline{i}) &= {}^+\lambda_i + 1 - i, & \text{if } 1 \leq i \leq k, \\ f_\lambda^{\mathbf{b}1}(i) &= f_\lambda^{\mathbf{b}}(i), & \text{if } i \in [m+n], \\ f_\lambda^{\mathbf{b}1}(\underline{i}) &= i - {}^+\lambda_i, & \text{if } 1 \leq i \leq k. \end{aligned}$$

The normalization $\rho_{\mathbf{b}}$ in (6.5) used in the definition of $f_\lambda^{\mathbf{b}} \in \mathbb{Z}^{m+n}$ in (6.7) is compatible with the above definitions in the sense that $f_\lambda^{\mathbf{b}0}$ and $f_\lambda^{\mathbf{b}1}$ correspond indeed to $\lambda + \rho$ for suitably normalized Weyl vector ρ for $\mathfrak{gl}(m+k|n)$ and $\mathfrak{gl}(m|n+k)$, respectively.

8.3. Classical KL theory. In this subsection we consider the case when $n = 0$ so that $\mathbf{b} = (0^m)$.

In this case, \mathfrak{g} defined in §7.1 and $\mathfrak{g}^k = \mathfrak{gl}(m+k)$ in §7.4 are Lie algebras. For $k \in \mathbb{N} \cup \{\infty\}$, recall the parabolic BGG category $\mathcal{O}_{\mathbf{b}}^k$ of \mathfrak{g}^k -modules defined in §7.2 and §7.4, and let $\mathcal{O}_{\mathbf{b}}^{k,\Delta}$ denote the full subcategory of $\mathcal{O}_{\mathbf{b}}^k$ consisting of objects that have finite parabolic \mathbf{b} -Verma flags (here and below it is understood that $\mathfrak{g}^\infty = \mathfrak{g}$, $\mathcal{O}_{\mathbf{b}}^\infty = \mathcal{O}_{\mathbf{b}}$, $\mathcal{O}_{\mathbf{b}}^{\infty,\Delta} = \mathcal{O}_{\mathbf{b}}^\Delta$ and so on). Let $[\mathcal{O}_{\mathbf{b}}^{k,\Delta}]$ denote its Grothendieck group.

Note that $\mathbb{T}^{\mathbf{b}} = \mathbb{V}^{\otimes m}$ for $\mathbf{b} = (0^m)$. Thanks to the bijection (8.1) given by $\lambda \mapsto f_\lambda^{\mathbf{b}0}$, the \mathbb{Z} -linear map

$$\Psi : [\mathcal{O}_{\mathbf{b}}^{k,\Delta}] \longrightarrow \mathbb{T}_{\mathbb{Z}}^{\mathbf{b}} \otimes \wedge^k \mathbb{V}_{\mathbb{Z}}, \quad [\mathcal{M}_{\mathbf{b}}^k(\lambda)] \mapsto M_{f_\lambda^{\mathbf{b}0}}^{\mathbf{b},0}(1),$$

is an isomorphism, where $\mathbb{T}_{\mathbb{Z}}^{\mathbf{b}} \otimes \wedge^k \mathbb{V}_{\mathbb{Z}}$ denotes the $q = 1$ specialization. The completion $[[\mathcal{O}_{\mathbf{b}}^{k,\Delta}]]$ of $[\mathcal{O}_{\mathbf{b}}^{k,\Delta}]$ is defined in exactly such a way so that Ψ extends to a \mathbb{Z} -linear isomorphism $\Psi : [[\mathcal{O}_{\mathbf{b}}^{k,\Delta}]] \longrightarrow \mathbb{T}_{\mathbb{Z}}^{\mathbf{b}} \hat{\otimes} \wedge^k \mathbb{V}_{\mathbb{Z}}$.

Then, by Vogan's homological interpretation of the Kazhdan-Lusztig polynomials [Vo, Conjecture 3.4] and [BGS, Theorem 3.11.4], Theorem 8.3 below (for k finite) is a well-known Fock space reformulation of the Kazhdan-Lusztig conjectures for type A Lie algebras [KL1, KL2] (proved in [BB, BK], and the equivalent tilting module version in [So2]). Such a reformulation can be found in [Br1, Br4] and [CW1, Theorem 4.14] (also see the proof of [CWZ, Theorem 5.4]). The case $k = \infty$ follows from the cases for finite k by Proposition 4.6, once the existence of tilting modules is established as in [CW1, Theorem 4.16]. Recall the KLV polynomials $\mathfrak{l}_{\mu\lambda}^{\mathbf{b},0}(q)$ from §7.4 and the BKL polynomials $\ell_{f_\mu^{\mathbf{b}0} f_\lambda^{\mathbf{b}0}}^{\mathbf{b},0}(q)$ from Proposition 4.5. Here we recall our assumption that $\mathbf{b} = (0^m)$ in the theorem below, though eventually it turns out to be valid for a general $0^m 1^n$ -sequence \mathbf{b} , and our formulation in this subsection makes sense for a general \mathbf{b} .

Theorem 8.3. *Let $k \in \mathbb{Z}_+ \cup \{\infty\}$. Then the isomorphism $\Psi : [[\mathcal{O}_{\mathbf{b}}^{k,\Delta}]] \longrightarrow \mathbb{T}_{\mathbb{Z}}^{\mathbf{b}} \hat{\otimes} \wedge^k \mathbb{V}_{\mathbb{Z}}$ satisfies*

$$\Psi([\mathcal{L}_{\mathbf{b}}^k(\lambda)]) = L_{f_\lambda^{\mathbf{b}0}}^{\mathbf{b},0}(1), \quad \Psi([\mathcal{T}_{\mathbf{b}}^k(\lambda)]) = T_{f_\lambda^{\mathbf{b}0}}^{\mathbf{b},0}(1), \quad \text{for } \lambda \in X^{k,+}.$$

Moreover, we have $\mathfrak{l}_{\mu\lambda}^{\mathbf{b},0}(q) = \ell_{f_\mu^{\mathbf{b}0} f_\lambda^{\mathbf{b}0}}^{\mathbf{b},0}(q)$, for $\lambda, \mu \in X^{k,+}$.

8.4. Super duality and BKL. Let \mathbf{b} be an arbitrary $0^m 1^n$ -sequence. Let us denote by $[[\check{\mathcal{O}}_{\mathbf{b}}^{\Delta}]]$ a similar completion of the Grothendieck group of the full subcategory $\check{\mathcal{O}}_{\mathbf{b}}^{\Delta}$ of $\check{\mathcal{O}}_{\mathbf{b}}$ consisting of objects with parabolic \mathbf{b} -Verma flags as above. Thanks to the bijection (8.1), we now have a \mathbb{Z} -linear isomorphism on the completions:

$$\check{\Psi} : [[\check{\mathcal{O}}_{\mathbf{b}}^{\Delta}]] \longrightarrow \mathbb{T}_{\mathbb{Z}}^{\mathbf{b}} \hat{\otimes} \wedge^{\infty} \mathbb{W}_{\mathbb{Z}}, \quad [\check{\mathcal{M}}_{\mathbf{b}}(\lambda)] \mapsto M_{f_{\lambda}^{\mathbf{b},1}}^{\mathbf{b},1}(1),$$

which is induced by the corresponding isomorphism $[\check{\mathcal{O}}_{\mathbf{b}}^{\Delta}] \cong \mathbb{T}_{\mathbb{Z}}^{\mathbf{b}} \otimes \wedge^{\infty} \mathbb{W}_{\mathbb{Z}}$. The following is a consequence of super duality (see Theorem 4.8 and Theorem 7.2).

Theorem 8.4. *Let \mathbf{b} be a $0^m 1^n$ -sequence. Assume the statement in Theorem 8.3 is valid for \mathbf{b} . Then, the isomorphism $\check{\Psi} : [[\check{\mathcal{O}}_{\mathbf{b}}^{\Delta}]] \longrightarrow \mathbb{T}_{\mathbb{Z}}^{\mathbf{b}} \hat{\otimes} \wedge^{\infty} \mathbb{W}_{\mathbb{Z}}$ satisfies*

$$\check{\Psi}([\check{\mathcal{L}}_{\mathbf{b}}^k(\lambda)]) = L_{f_{\lambda}^{\mathbf{b},1}}^{\mathbf{b},1}(1), \quad \check{\Psi}([\check{\mathcal{T}}_{\mathbf{b}}^k(\lambda)]) = T_{f_{\lambda}^{\mathbf{b},1}}^{\mathbf{b},1}(1), \quad \text{for } \lambda \in \check{X}^+.$$

Consequently, we have $\check{\ell}_{\mu\lambda}^{\mathbf{b}}(q) = \ell_{f_{\mu}^{\mathbf{b},1} f_{\lambda}^{\mathbf{b},1}}^{\mathbf{b},1}(q)$, for $\lambda, \mu \in \check{X}^+$.

Proof. The bijections given in (8.1) are compatible with the bijection \natural in (4.4), so $\natural(f_{\lambda}^{\mathbf{b},0}) = f_{\lambda^{\natural}}^{\mathbf{b},1}$. Combining the isomorphism $\natural_{\mathbf{b}} : \mathbb{T}^{\mathbf{b}} \hat{\otimes} \wedge^{\infty} \mathbb{V} \rightarrow \mathbb{T}^{\mathbf{b}} \hat{\otimes} \wedge^{\infty} \mathbb{W}$ from Theorem 4.8, super duality (SD) from Theorem 7.2, and the isomorphism Ψ from Theorem 8.3, we have the following commutative diagram of isomorphisms on the left (the maps are defined in terms of the standard objects on the right diagram):

$$(8.2) \quad \begin{array}{ccc} [[\mathcal{O}_{\mathbf{b}}^{\Delta}]] & \xrightarrow{\Psi} & \mathbb{T}_{\mathbb{Z}}^{\mathbf{b}} \hat{\otimes} \wedge^{\infty} \mathbb{V}_{\mathbb{Z}} \\ \text{SD} \downarrow & & \natural_{\mathbf{b}} \downarrow \\ [[\check{\mathcal{O}}_{\mathbf{b}}^{\Delta}]] & \xrightarrow{\check{\Psi}} & \mathbb{T}_{\mathbb{Z}}^{\mathbf{b}} \hat{\otimes} \wedge^{\infty} \mathbb{W}_{\mathbb{Z}} \end{array} \quad \begin{array}{ccc} [\mathcal{M}_{\mathbf{b}}(\lambda)] & \xrightarrow{\Psi} & M_{f_{\lambda}^{\mathbf{b},0}}^{\mathbf{b},0}(1) \\ \text{SD} \downarrow & & \natural_{\mathbf{b}} \downarrow \\ [\check{\mathcal{M}}_{\mathbf{b}}(\lambda^{\natural})] & \xrightarrow{\check{\Psi}} & M_{f_{\lambda^{\natural}}^{\mathbf{b},1}}^{\mathbf{b},1}(1) \end{array}$$

Note that $\natural_{\mathbf{b}}$ preserves the (dual) canonical bases by Theorem 4.8(3), super duality SD preserves the simple and tilting modules by Theorem 7.2(2), and Ψ preserves the L 's and T 's by assumption that the statement in Theorem 8.3 is valid. Therefore, we have established the three sides (except the arrow on $\check{\Psi}$) in the following diagrams:

$$\begin{array}{ccc} [\mathcal{L}_{\mathbf{b}}(\lambda)] & \xrightarrow{\Psi} & L_{f_{\lambda}^{\mathbf{b},0}}^{\mathbf{b},0}(1) \\ \text{SD} \downarrow & & \natural_{\mathbf{b}} \downarrow \\ [\check{\mathcal{L}}_{\mathbf{b}}(\lambda^{\natural})] & \xrightarrow{\check{\Psi}} & L_{f_{\lambda^{\natural}}^{\mathbf{b},1}}^{\mathbf{b},1}(1) \end{array} \quad \begin{array}{ccc} [\mathcal{T}_{\mathbf{b}}(\lambda)] & \xrightarrow{\Psi} & T_{f_{\lambda}^{\mathbf{b},0}}^{\mathbf{b},0}(1) \\ \text{SD} \downarrow & & \natural_{\mathbf{b}} \downarrow \\ [\check{\mathcal{T}}_{\mathbf{b}}(\lambda^{\natural})] & \xrightarrow{\check{\Psi}} & T_{f_{\lambda^{\natural}}^{\mathbf{b},1}}^{\mathbf{b},1}(1) \end{array}$$

The arrows for the map $\check{\Psi}$ in ? now follow from the commutativity of (8.2).

The identity $\check{\ell}_{\mu\lambda}^{\mathbf{b}}(q) = \ell_{f_{\mu}^{\mathbf{b},1} f_{\lambda}^{\mathbf{b},1}}^{\mathbf{b},1}(q)$ follows by the corresponding identities in Theorem 4.8(4), Theorem 7.2(3), and Theorem 8.3. \square

8.5. Comparison of characters. Let \mathbf{b} be a fixed $0^m 1^n$ -sequence. For $k \in \mathbb{N}$ let $(\mathbf{b}, 0^k)$ and $(\mathbf{b}, 1^k)$ denote the $0^{m+k} 1^n$ - and the $0^m 1^{n+k}$ -sequences obtained by adding k 0's and k 1's to the end of the sequence \mathbf{b} , respectively. Recall from §6.3 that $\mathcal{O}_{(\mathbf{b}, 0^k)}^{m+k|n}$ and $\mathcal{O}_{(\mathbf{b}, 1^k)}^{m|n+k}$ are the full BGG categories of $\mathfrak{gl}(m+k|n)$ - and $\mathfrak{gl}(m|n+k)$ -modules, respectively. In this subsection, we compare the simple modules as well as tilting modules in the parabolic category $\mathcal{O}_{\mathbf{b}}^k$ (and respectively, $\check{\mathcal{O}}_{\mathbf{b}}^k$) introduced in §7.4 with their counterparts in the full BGG category $\mathcal{O}_{(\mathbf{b}, 0^k)}^{m+k|n}$ (and respectively, $\mathcal{O}_{(\mathbf{b}, 1^k)}^{m|n+k}$). Also, note by (7.7) and (7.8) that $X^{k,+} \subseteq X^k$, $\check{X}^{k,+} \subseteq \check{X}^k$.

For $\lambda \in X^{k,+}$, we can express $[L_{(\mathbf{b}, 0^k)}(\lambda)]$ in terms of Verma modules:

$$(8.3) \quad [L_{(\mathbf{b}, 0^k)}(\lambda)] = \sum_{\mu \in X^k} a_{\mu\lambda} [M_{(\mathbf{b}, 0^k)}(\mu)], \quad \text{for } a_{\mu\lambda} \in \mathbb{Z}.$$

Since the simple objects in the parabolic category $\mathcal{O}_{\mathbf{b}}^k$ defined in §7.4 are the $\mathfrak{g}^k \equiv \mathfrak{gl}(m+k|n)$ -modules $\mathcal{L}_{\mathbf{b}}^k(\lambda) \equiv L_{(\mathbf{b}, 0^k)}(\lambda)$, for $\lambda \in X^{k,+}$, we can also express $[L_{(\mathbf{b}, 0^k)}(\lambda)]$ in terms of *parabolic* Verma modules in $\mathcal{O}_{\mathbf{b}}^k$:

$$(8.4) \quad [L_{(\mathbf{b}, 0^k)}(\lambda)] = \sum_{\nu \in X^{k,+}} b_{\nu\lambda} [\mathcal{M}_{\mathbf{b}}^k(\nu)], \quad \text{for } b_{\nu\lambda} \in \mathbb{Z}.$$

Here we recall that $\mathcal{M}_{\mathbf{b}}^k(\nu) = \text{Ind}_{\mathfrak{p}_{\mathbf{b}}^k}^{\mathfrak{g}^k} L^0(\nu)$, where $L^0(\nu)$ is the irreducible $(\mathfrak{h}^k + \mathfrak{gl}(k))$ -module of highest weight $\nu \in X^{k,+}$. Applying the Weyl character formula to $L^0(\nu)$ gives us $[\mathcal{M}_{\mathbf{b}}^k(\nu)] = \sum_{\sigma \in \mathfrak{S}_k} (-1)^{\ell(\sigma)} [M_{(\mathbf{b}, 0^k)}(\sigma \cdot \nu)]$, where as usual we have denoted the dot action of a Weyl group element σ on ν by $\sigma \cdot \nu = \sigma(\nu + \rho_{(\mathbf{b}, 0^k)}) - \rho_{(\mathbf{b}, 0^k)}$ and the Weyl vector $\rho_{(\mathbf{b}, 0^k)}$ is defined in §6.4. Hence, (8.4) can be rewritten as

$$[L_{(\mathbf{b}, 0^k)}(\lambda)] = \sum_{\nu \in X^{k,+}} \sum_{\sigma \in \mathfrak{S}_k} (-1)^{\ell(\sigma)} b_{\nu\lambda} [M_{(\mathbf{b}, 0^k)}(\sigma \cdot \nu)], \quad \text{for } \lambda \in X^{k,+}.$$

Comparing this with (8.3) together with the linear independence of the Verma characters show that $a_{\nu\lambda} = b_{\nu\lambda}$, for $\lambda, \nu \in X^{k,+}$. We summarize this in the following.

Proposition 8.5. *Let $\lambda \in X^{k,+}$. Retain the notation as in (8.3). Then*

$$[L_{(\mathbf{b}, 0^k)}(\lambda)] = \sum_{\nu \in X^{k,+}} a_{\nu\lambda} [\mathcal{M}_{\mathbf{b}}^k(\nu)].$$

Similarly, the simple $\mathfrak{gl}(m|n+k)$ -module $\check{\mathcal{L}}_{\mathbf{b}}^k(\xi)$, for $\xi \in \check{X}^{k,+}$, in the parabolic category $\check{\mathcal{O}}_{\mathbf{b}}^k$ (cf. §7.4) can be identified with $L_{(\mathbf{b}, 1^k)}(\xi)$ in the full BGG category $\mathcal{O}_{(\mathbf{b}, 1^k)}^{m|n+k}$. We write that

$$(8.5) \quad [L_{(\mathbf{b}, 1^k)}(\xi)] = \sum_{\mu \in \check{X}^k} \check{a}_{\eta\xi} [M_{(\mathbf{b}, 1^k)}(\eta)], \quad \text{for } \check{a}_{\eta\xi} \in \mathbb{Z}, \xi \in \check{X}^{k,+}.$$

By a parallel argument as above, we obtain the following.

Proposition 8.6. *Let $\xi \in \check{X}^{k,+}$. Retain the notation as in (8.5). Then*

$$[L_{(\mathbf{b},1^k)}(\xi)] = \sum_{\eta \in \check{X}^{k,+}} \check{a}_{\eta\xi} [\check{\mathcal{M}}_{\mathbf{b}}^k(\eta)].$$

We now proceed to compare the characters of the tilting modules in a parabolic BGG category with those in a full BGG category. In the full BGG categories $\mathcal{O}_{(\mathbf{b},0^k)}^{m+k|n}$ and $\mathcal{O}_{(\mathbf{b},1^k)}^{m|n+k}$, we write the following.

$$(8.6) \quad [T_{(\mathbf{b},0^k)}(\lambda)] = \sum_{\mu \in X^k} c_{\mu\lambda} [M_{(\mathbf{b},0^k)}(\mu)], \quad \text{for } c_{\mu\lambda} \in \mathbb{Z}, \lambda \in X^{k,+};$$

$$(8.7) \quad [T_{(\mathbf{b},1^k)}(\xi)] = \sum_{\eta \in \check{X}^k} \check{c}_{\eta\xi} [M_{(\mathbf{b},1^k)}(\eta)], \quad \text{for } \check{c}_{\eta\xi} \in \mathbb{Z}, \xi \in \check{X}^{k,+}.$$

Recall the tilting modules $\mathcal{T}_{\mathbf{b}}^k(\lambda)$, for $\lambda \in X^{k,+}$, in the parabolic category $\mathcal{O}_{\mathbf{b}}^k$, and the tilting modules $\check{\mathcal{T}}_{\mathbf{b}}^k(\xi)$ in $\check{\mathcal{O}}_{\mathbf{b}}^k$, for $\xi \in \check{X}^{k,+}$.

Proposition 8.7. (1) *Let $\lambda \in X^{k,+}$. Retain the notation in (8.6), and write*

$$[\mathcal{T}_{\mathbf{b}}^k(\lambda)] = \sum_{\nu \in X^{k,+}} d_{\nu\lambda} [\mathcal{M}_{\mathbf{b}}^k(\nu)].$$

Then $d_{\nu\lambda} = \sum_{\tau \in \mathfrak{S}_k} (-1)^{\ell(\tau w_0)} c_{\tau \cdot \nu, w_0 \cdot \lambda}$.

(2) *Let $\xi \in \check{X}^{k,+}$. Retain the notation in (8.7), and write*

$$[\check{\mathcal{T}}_{\mathbf{b}}^k(\xi)] = \sum_{\eta \in \check{X}^{k,+}} \check{d}_{\eta\xi} [\check{\mathcal{M}}_{\mathbf{b}}^k(\eta)].$$

Then $\check{d}_{\eta\xi} = \sum_{\tau \in \mathfrak{S}_k} (-1)^{\ell(\tau w_0)} \check{c}_{\tau \cdot \eta, w_0 \cdot \xi}$.

Proof. We shall only prove (1), as (2) is analogous.

Set $\rho = \rho_{(\mathbf{b},0^k)}$, $\rho_{\mathbf{u}} = \frac{1}{2} \sum_{\alpha \in \mathbf{u}_{\mathbf{b}}^k} (-1)^{|\alpha|} \alpha$, and $\rho_{\mathfrak{t}} = \rho - \rho_{\mathbf{u}}$, where $|\alpha|$ denotes the parity of the root α . Furthermore, let $w_0 = w_0^{(k)}$ be the longest element in \mathfrak{S}_k . Applying [So2, Theorem 6.7] and its super generalization [Br2, Theorem 6.4] to the category $\mathcal{O}_{(\mathbf{b},0^k)}^{m+k|n}$, we compute, for $\lambda, \mu \in X^{k,+}$,

$$\begin{aligned} [\mathcal{M}_{\mathbf{b}}^k(\lambda) : L_{(\mathbf{b},0^k)}(\mu)] &= \sum_{\tau \in \mathfrak{S}_k} (-1)^{\ell(\tau)} [M_{(\mathbf{b},0^k)}(\tau \cdot \lambda) : L_{(\mathbf{b},0^k)}(\mu)] \\ &= \sum_{\tau \in \mathfrak{S}_k} (-1)^{\ell(\tau)} \left(T_{(\mathbf{b},0^k)}(-2\rho - \mu) : M_{(\mathbf{b},0^k)}(-2\rho - \tau \cdot \lambda) \right) \\ &= \sum_{\tau \in \mathfrak{S}_k} (-1)^{\ell(\tau)} c_{-2\rho - \tau \cdot \lambda, -2\rho - \mu}. \end{aligned}$$

On the other hand, by applying [Br2, Theorem 6.4] to the category $\mathcal{O}_{\mathbf{b}}^k$, we also have

$$\begin{aligned} \left[\mathcal{M}_{\mathbf{b}}^k(\lambda) : L_{(\mathbf{b}, 0^k)}(\mu) \right] &= \left(\mathcal{T}_{\mathbf{b}}^k(-2\rho_{\mathbf{u}} - w_0\mu) : \mathcal{M}_{\mathbf{b}}^k(-2\rho_{\mathbf{u}} - w_0\lambda) \right) \\ &= d_{-2\rho_{\mathbf{u}} - w_0\lambda, -2\rho_{\mathbf{u}} - w_0\mu}. \end{aligned}$$

A comparison of the above two identities and replacing τ by τw_0 give us

$$(8.8) \quad d_{-2\rho_{\mathbf{u}} - w_0\lambda, -2\rho_{\mathbf{u}} - w_0\mu} = \sum_{\tau \in \mathfrak{S}_k} (-1)^{\ell(\tau w_0)} c_{-2\rho - \tau w_0 \cdot \lambda, -2\rho - \mu}.$$

Set $\nu = -2\rho_{\mathbf{u}} - w_0\lambda$ and $\eta = -2\rho_{\mathbf{u}} - w_0\mu$. We shall use repeatedly the following simple identities:

$$\rho = \rho_{\mathfrak{t}} + \rho_{\mathbf{u}}, \quad \tau\rho_{\mathbf{u}} = w_0\rho_{\mathbf{u}} = \rho_{\mathbf{u}}, \quad w_0\rho_{\mathfrak{t}} = -\rho_{\mathfrak{t}}.$$

Now we compute

$$\begin{aligned} -2\rho - \tau w_0 \cdot \lambda &= -\rho - \tau w_0(\lambda + \rho) = -\rho_{\mathbf{u}} - \tau w_0\lambda - \tau w_0\rho_{\mathfrak{t}} - \rho \\ &= \tau(-2\rho_{\mathbf{u}} - w_0\lambda + \rho) - \rho = \tau \cdot \nu. \end{aligned}$$

Also, $w_0 \cdot \eta = w_0(-2\rho_{\mathbf{u}} - w_0\mu + \rho) - \rho = -\rho_{\mathbf{u}} - \mu - \rho_{\mathfrak{t}} - \rho = -2\rho - \mu$. From these computations, we see that (8.8) rewritten in terms of ν and η (then followed by a change of notation η to λ) is exactly what we want to prove in (1). \square

8.6. BKL for adjacent Borel subalgebras. The following theorem is a key step in our proof of the BKL Conjecture 8.1.

Theorem 8.8. *Let \mathbf{b} and \mathbf{b}' be two adjacent $0^m 1^n$ -sequences. The BKL Conjecture 8.1 holds for \mathbf{b} if and only if it holds for \mathbf{b}' .*

Proof. It suffices to prove that the validity of BKL Conjecture 8.1 for \mathbf{b} implies its validity for \mathbf{b}' . We shall follow the notations in §5.2 to denote $\mathbf{b} = (\mathbf{b}^1, 0, 1, \mathbf{b}^2)$ and $\mathbf{b}' := (\mathbf{b}^1, 1, 0, \mathbf{b}^2)$. (The proof below goes through similarly when switching \mathbf{b} and \mathbf{b}' .)

(1) We first prove this for Part (1) of BKL Conjecture 8.1. The idea of the proof is to switch to the bases in notation N 's instead of the M 's for more effective comparisons with the L 's, based on the results of §5.2, §5.3 and Section 6.

Recall $\psi_{\mathbf{b}}([M_{\mathbf{b}}(\lambda)]) = M_{f_{\lambda}^{\mathbf{b}}}(1)$, for all λ . By comparing (5.6) and (6.15), we have

$$(8.9) \quad \psi_{\mathbf{b}}([N_{\mathbf{b}}(\mu)]) = N_{f_{\mu}^{\mathbf{b}}}(1), \quad \text{for } \mu \in X(m|n).$$

By the assumption on the validity of the BKL Conjecture 8.1(1) for \mathbf{b} , we have

$$(8.10) \quad \psi_{\mathbf{b}}([L_{\mathbf{b}}(\lambda)]) = L_{f_{\lambda}^{\mathbf{b}}}(1), \quad \text{for } \lambda \in X(m|n).$$

By (5.9) and Proposition 5.8, we have

$$(8.11) \quad L_f^{\mathbf{b}} = \sum_g \check{\ell}_{gf}(q) N_g.$$

Since $\psi_{\mathbf{b}}$ is an isomorphism, it follows by (8.9), (8.10) and (8.11) that

$$(8.12) \quad [L_{\mathbf{b}}(\lambda)] = \sum_{\mu} \check{\ell}_{f_{\mu}^{\mathbf{b}} f_{\lambda}^{\mathbf{b}}}(1) [N_{\mathbf{b}}(\mu)].$$

Lemma 6.2 states that $L_{\mathbf{b}}(\lambda) = L_{\mathbf{b}'}(\lambda^{\mathbb{L}})$, (6.14) states that $\text{ch}N_{\mathbf{b}}(\mu) = \text{ch}N_{\mathbf{b}'}(\mu^{\mathbb{L}})$, while Corollary 5.13 states that $\check{\ell}_{gf}(q) = \check{\ell}'_{g^{\mathbb{L}}f^{\mathbb{L}}}(q)$. These three identities together with (8.12) imply that

$$(8.13) \quad [L_{\mathbf{b}'}(\lambda^{\mathbb{L}})] = \sum_{\mu} \check{\ell}'_{f_{\mu^{\mathbb{L}}}f_{\lambda^{\mathbb{L}}}^{\mathbf{b}'}}(1) [N_{\mathbf{b}'}(\mu^{\mathbb{L}})],$$

where we have identified $f_{\mu^{\mathbb{L}}}^{\mathbf{b}'} = (f_{\mu}^{\mathbf{b}})^{\mathbb{L}}$ for all μ by the definitions of (5.16) and (6.1).

By Proposition 5.10 and (5.26), we have

$$(8.14) \quad L_f^{\mathbf{b}'} = \sum_g \check{\ell}'_{gf}(q) N'_g.$$

By definition, $\psi_{\mathbf{b}'}([M_{\mathbf{b}'}(\lambda)]) = M_{f_{\lambda}^{\mathbf{b}'}}(1)$. By straightforward \mathbf{b}' -counterparts of (5.6) and (6.15), we have the following \mathbf{b}' -counterpart of (8.9):

$$(8.15) \quad \psi_{\mathbf{b}'}([N_{\mathbf{b}'}(\mu^{\mathbb{L}})]) = N'_{f_{\mu^{\mathbb{L}}}^{\mathbf{b}'}}(1), \quad \text{for } \mu \in X(m|n).$$

Now applying $\psi_{\mathbf{b}'}$ to both sides of (8.13) and using (8.15), we obtain by a comparison with (8.14) that $\psi_{\mathbf{b}'}([L_{\mathbf{b}'}(\lambda^{\mathbb{L}})]) = L_{\lambda^{\mathbb{L}}}^{\mathbf{b}'}(1)$. This proves the BKL Conjecture 8.1(1) for \mathbf{b}' .

(2) We employ a similar strategy to prove that the validity of BKL Conjecture 8.1(2) for \mathbf{b} implies its validity for \mathbf{b}' . The idea of the proof is to switch to the bases in notation U 's instead of the M 's for more effective comparisons with the T 's, based on the results of §5.2, §5.3 and Section 6.

By comparing (5.11) and (6.16), we have

$$(8.16) \quad \psi_{\mathbf{b}}([U_{\mathbf{b}}(\mu)]) = U_{f_{\mu}^{\mathbf{b}}} (1), \quad \text{for } \mu \in X(m|n).$$

By the assumption on the validity of the BKL Conjecture 8.1(2) for \mathbf{b} , we have

$$(8.17) \quad \psi_{\mathbf{b}}([T_{\mathbf{b}}(\lambda)]) = T_{f_{\lambda}^{\mathbf{b}}}^{\mathbf{b}}(1), \quad \text{for } \lambda \in X(m|n).$$

By (5.13) and Proposition 5.8, we have

$$(8.18) \quad T_f^{\mathbf{b}} = \sum_g \check{t}_{gf}(q) U_g.$$

Since $\psi_{\mathbf{b}}$ is an isomorphism, it follows by (8.16), (8.17) and (8.18) that

$$(8.19) \quad [T_{\mathbf{b}}(\lambda)] = \sum_{\mu} \check{t}_{f_{\mu}^{\mathbf{b}}f_{\lambda}^{\mathbf{b}}} (1) [U_{\mathbf{b}}(\mu)].$$

Theorem 6.10 states that $T_{\mathbf{b}}(\lambda) = T_{\mathbf{b}'}(\lambda^{\mathbb{U}})$, (6.14) states that $\text{ch}U_{\mathbf{b}}(\mu) = \text{ch}U_{\mathbf{b}'}(\mu^{\mathbb{U}})$, while Corollary 5.15 states that $\check{t}_{gf}(q) = \check{t}'_{g^{\mathbb{U}}f^{\mathbb{U}}}(q)$. These three identities together with (8.19) imply that

$$(8.20) \quad [T_{\mathbf{b}'}(\lambda^{\mathbb{U}})] = \sum_{\mu} \check{t}'_{f_{\mu^{\mathbb{U}}}f_{\lambda^{\mathbb{U}}}^{\mathbf{b}'}}(1) [U_{\mathbf{b}'}(\mu^{\mathbb{U}})],$$

where we have identified $f_{\mu^{\mathbb{U}}}^{\mathbf{b}'} = (f_{\mu}^{\mathbf{b}})^{\mathbb{U}}$ for all μ by the definitions of (5.17) and (6.8).

By Proposition 5.10 and (5.28), we have

$$(8.21) \quad T_f^{\mathbf{b}'} = \sum_g \check{t}_{gf}'(q) U_g'.$$

By definition, $\psi_{\mathbf{b}'}([M_{\mathbf{b}'}(\lambda)]) = M_{f_{\lambda}^{\mathbf{b}'}}(1)$. We easily have the following \mathbf{b}' -counterpart of (8.16):

$$(8.22) \quad \psi_{\mathbf{b}'}([U_{\mathbf{b}'}(\mu^{\mathbb{U}})]) = U_{f_{\mu^{\mathbb{U}}}^{\mathbf{b}'}}'(1), \quad \text{for } \mu \in X(m|n).$$

Now applying $\psi_{\mathbf{b}'}$ to both sides of (8.20) and using (8.22), we obtain by a comparison with (8.21) that $\psi_{\mathbf{b}'}([T_{\mathbf{b}'}(\lambda^{\mathbb{U}})]) = T_{f_{\lambda^{\mathbb{U}}}^{\mathbf{b}'}}^{\mathbf{b}'}(1)$. This proves BKL Conjecture 8.1(2) for \mathbf{b}' .

The proof of the theorem is completed. \square

Remark 8.9. Any two $0^m 1^n$ -sequences are connected via a sequence of $0^m 1^n$ -sequences such that any two neighboring sequences are adjacent. Therefore, Theorem 8.8 is equivalent to saying that the validity of the BKL Conjecture 8.1 for *one* particular $0^m 1^n$ -sequence implies its validity for *all* $0^m 1^n$ -sequences.

8.7. The proof. The following theorem will provide the inductive step for proving the BKL conjecture. We follow the outline of steps (1.2)–(1.5) in the Introduction.

Theorem 8.10. *Let $n \geq 0$ be fixed. The validity of the BKL conjecture for all $0^m 1^n$ -sequences for every $m \geq 0$ implies the validity of the BKL conjecture for some $0^m 1^{n+1}$ -sequence for every m .*

Proof. In this proof, we will regard m and n as fixed, and prove the following reformulation: the validity of the BKL conjecture for all $0^{m+k} 1^n$ -sequences for every $k \geq 0$ implies the validity of the BKL conjecture for one particular $0^m 1^{n+1}$ -sequence.

Take an arbitrary $0^m 1^n$ -sequence \mathbf{b} , and form the $0^{m+k} 1^n$ -sequence $(\mathbf{b}, 0^k)$. The assumption above that the BKL conjecture for all $0^{m+k} 1^n$ -sequences for every k holds can be stated more precisely as follows.

(A) The isomorphism

$$[[\mathcal{O}_{(\mathbf{b}, 0^k)}^{m+k|n, \Delta}]] \longrightarrow \widehat{\mathbb{T}}_{\mathbb{Z}}^{(\mathbf{b}, 0^k)}, \quad [M_{(\mathbf{b}, 0^k)}(\lambda)] \mapsto M_{f_{\lambda}^{(\mathbf{b}, 0^k)}}^{(\mathbf{b}, 0^k)}(1),$$

sends $[L_{(\mathbf{b}, 0^k)}(\lambda)]$ to $L_{f_{\lambda}^{(\mathbf{b}, 0^k)}}^{(\mathbf{b}, 0^k)}(1)$, and $[T_{(\mathbf{b}, 0^k)}(\lambda)]$ to $T_{f_{\lambda}^{(\mathbf{b}, 0^k)}}^{(\mathbf{b}, 0^k)}(1)$, for all $\lambda \in X(m+k|n)$.

We proceed in 4 steps (i)–(iv) below, starting from (A). Note that $\mathbb{T}^{(\mathbf{b}, 0^k)} = \mathbb{T}^{\mathbf{b}} \otimes \mathbb{V}^{\otimes k}$.

(i) Pass to a parabolic version.

The isomorphism

$$[[\mathcal{O}_{\mathbf{b}}^{k, \Delta}]] \longrightarrow \mathbb{T}_{\mathbb{Z}}^{\mathbf{b}} \widehat{\otimes} \wedge^k \mathbb{V}_{\mathbb{Z}}, \quad [\mathcal{M}_{\mathbf{b}}^k(\lambda)] \mapsto M_{f_{\lambda}^{\mathbf{b}, 0}}^{\mathbf{b}, 0}(1),$$

sends $[\mathcal{L}_{\mathbf{b}}^k(\lambda)]$ to $L_{f_{\lambda}^{\mathbf{b}, 0}}^{\mathbf{b}, 0}(1)$, and $[\mathcal{T}_{\mathbf{b}}^k(\lambda)]$ to $T_{f_{\lambda}^{\mathbf{b}, 0}}^{\mathbf{b}, 0}(1)$ for $\lambda \in X^{k, +}$.

Indeed (i) follows from (A), by Propositions 4.9 and 4.10 (which relate the BKL polynomials from the setting of (A) to the current q -wedge setting), as well as Propositions 8.5 and 8.7 (which relate the simple and tilting modules from the setting of (A) to the current parabolic setting).

(ii) Pass from finite k to ∞ .

The isomorphism

$$[[\mathcal{O}_{\mathbf{b}}^{\Delta}]] \longrightarrow \mathbb{T}_{\mathbb{Z}}^{\mathbf{b}} \widehat{\otimes} \wedge^{\infty} \mathbb{V}_{\mathbb{Z}}, \quad [\mathcal{M}_{\mathbf{b}}(\lambda)] \mapsto M_{f_{\lambda}^{\mathbf{b},0}}^{\mathbf{b},0}(1),$$

sends $[\mathcal{L}_{\mathbf{b}}(\lambda)]$ to $L_{f_{\lambda}^{\mathbf{b},0}}^{\mathbf{b},0}(1)$, and $[\mathcal{T}_{\mathbf{b}}(\lambda)]$ to $T_{f_{\lambda}^{\mathbf{b},0}}^{\mathbf{b},0}(1)$ for $\lambda \in X^+$.

Indeed (ii) follows from (i) by Proposition 4.6 and Proposition 7.5.

(iii) Super duality.

Note that (i) and (ii) are exactly the statements formulated in Theorem 8.3, now valid for a general \mathbf{b} . Hence, the assumption in Theorem 8.4 is now valid. By Theorem 8.4, the isomorphism

$$\check{\Psi} : [[\check{\mathcal{O}}_{\mathbf{b}}^{\Delta}]] \longrightarrow \mathbb{T}_{\mathbb{Z}}^{\mathbf{b}} \widehat{\otimes} \wedge^{\infty} \mathbb{W}_{\mathbb{Z}}, \quad [\check{\mathcal{M}}_{\mathbf{b}}(\lambda)] \mapsto M_{f_{\lambda}^{\mathbf{b},1}}^{\mathbf{b},1}(1),$$

sends $[\check{\mathcal{L}}_{\mathbf{b}}(\lambda)]$ to $L_{f_{\lambda}^{\mathbf{b},1}}^{\mathbf{b},1}(1)$, and $[\check{\mathcal{T}}_{\mathbf{b}}(\lambda)]$ to $T_{f_{\lambda}^{\mathbf{b},1}}^{\mathbf{b},1}(1)$, for $\lambda \in \check{X}^+$.

(iv) Truncation.

Now let $k \in \mathbb{N}$. Recall the category $\check{\mathcal{O}}_{\mathbf{b}}^{k,\Delta}$ of $\mathfrak{gl}(m|n+k)$ -modules from §7.4, and recall the bijection $X^{k,+} \rightarrow \mathbb{Z}^{m+n} \times \mathbb{Z}_+^k$ by sending λ to $f_{\lambda}^{\mathbf{b},1}$ from (8.1). Consider the isomorphism

$$\check{\Psi}^k : [[\check{\mathcal{O}}_{\mathbf{b}}^{k,\Delta}]] \longrightarrow \mathbb{T}_{\mathbb{Z}}^{\mathbf{b}} \widehat{\otimes} \wedge^k \mathbb{W}_{\mathbb{Z}}, \quad [\check{\mathcal{M}}_{\mathbf{b}}^k(\lambda)] \mapsto M_{f_{\lambda}^{\mathbf{b},1}}^{\mathbf{b},1}(1),$$

where $[[\check{\mathcal{O}}_{\mathbf{b}}^{k,\Delta}]]$ is a suitable completion of $[\check{\mathcal{O}}_{\mathbf{b}}^{k,\Delta}]$ as before. We have the following.

Claim. For $\lambda \in \check{X}^{k,+}$, we have

$$(8.23) \quad \check{\Psi}^k([\check{\mathcal{L}}_{\mathbf{b}}^k(\lambda)]) = L_{f_{\lambda}^{\mathbf{b},1}}^{\mathbf{b},1}(1), \quad \check{\Psi}^k([\check{\mathcal{T}}_{\mathbf{b}}^k(\lambda)]) = T_{f_{\lambda}^{\mathbf{b},1}}^{\mathbf{b},1}(1).$$

Let us specialize $k = 1$. In this case, the parabolic category $\check{\mathcal{O}}_{\mathbf{b}}^1$ is exactly the full BGG category $\mathcal{O}^{m|n+1}$, and $\check{X}^{1,+} = X(m|n+1)$. Hence assuming the claim, we have verified the BKL conjecture for the special $0^m 1^{n+1}$ -sequence $(\mathbf{b}, 1)$.

It remains to prove (8.23) for $\lambda \in \check{X}^{k,+}$, using the truncation maps and truncation functors. We have the following commutative diagram by a direct computation using the basis $\{[\mathcal{M}_{\mathbf{b}}(\lambda)]\}$ for $[[\check{\mathcal{O}}_{\mathbf{b}}^{\Delta}]]$:

$$(8.24) \quad \begin{array}{ccc} [[\check{\mathcal{O}}_{\mathbf{b}}^{\Delta}]] & \xrightarrow{\check{\Psi}} & \mathbb{T}_{\mathbb{Z}}^{\mathbf{b}} \widehat{\otimes} \wedge^{\infty} \mathbb{W}_{\mathbb{Z}} \\ \check{\text{tr}} \downarrow & & \text{Tr} \downarrow \\ [[\check{\mathcal{O}}_{\mathbf{b}}^{k,\Delta}]] & \xrightarrow{\check{\Psi}^k} & \mathbb{T}_{\mathbb{Z}}^{\mathbf{b}} \widehat{\otimes} \wedge^k \mathbb{W}_{\mathbb{Z}} \end{array}$$

It follows from (iii), (8.24), Propositions 4.4 and 7.5 that (8.23) holds for those $\lambda \in \check{X}^{k,+}$ satisfying the condition $\langle \lambda, e_{m+n+k, m+n+k}^{(\mathbf{b}, 1^k)} \rangle \geq 0$. This condition arises in the parametrization set for the standard basis of the image of $\check{\text{tr}}$ (which is not surjective); see Remark 7.6.

We have the following commutative diagram:

$$(8.25) \quad \begin{array}{ccc} [[\check{\mathcal{O}}_{\mathbf{b}}^{k,\Delta}]] & \xrightarrow{\check{\Psi}^k} & \mathbb{T}_{\mathbb{Z}}^{\mathbf{b}} \hat{\otimes} \wedge^k \mathbb{W}_{\mathbb{Z}} \\ \otimes \text{Str} \downarrow & & \text{sh} \downarrow \\ [[\check{\mathcal{O}}_{\mathbf{b}}^{k,\Delta}]] & \xrightarrow{\check{\Psi}^k} & \mathbb{T}_{\mathbb{Z}}^{\mathbf{b}} \hat{\otimes} \wedge^k \mathbb{W}_{\mathbb{Z}} \end{array}$$

Here $\otimes \text{Str}$ denotes the map induced from tensoring with the 1-dimensional supertrace representation (see (6.3) with n therein replaced by $n+k$), and sh denotes the \mathbb{Z} -linear shift map which sends $M_f^{\mathbf{b},1}$ to $M_{f+1_{m|(n+k)}}^{\mathbf{b},1}$, for each f ; see (3.12) for notation $1_{m|(n+k)}$.

Just as in the proof of Proposition 3.11 where a similar shift map has been used, sh also commutes with the bar map, and then

$$(8.26) \quad \text{sh}(T_f^{\mathbf{b},1}) = T_{f+1_{m|(n+k)}}^{\mathbf{b},1}, \quad \text{sh}(L_f^{\mathbf{b},1}) = L_{f+1_{m|(n+k)}}^{\mathbf{b},1}, \quad \forall f.$$

On the other hand, it is clear that

$$(8.27) \quad \check{\mathcal{L}}_{\mathbf{b}}^k(\lambda) \otimes \text{Str} = \check{\mathcal{L}}_{\mathbf{b}}^k(\lambda + \text{Str}), \quad \check{\mathcal{T}}_{\mathbf{b}}^k(\lambda) \otimes \text{Str} = \check{\mathcal{T}}_{\mathbf{b}}^k(\lambda + \text{Str}), \quad \forall \lambda \in \check{X}^{k,+}.$$

It follows by (8.26), (8.27) and the commutative diagram (8.25) that (8.23) holds for $\lambda \in \check{X}^{k,+}$ satisfying $\langle \lambda, e_{m+n+k, m+n+k}^{(\mathbf{b},1^k)} \rangle \geq -1$. Repeatedly using (8.25), (8.26) and (8.27), we conclude that (8.23) holds for all $\lambda \in \check{X}^{k,+}$.

This proves the claim, and hence completes the proof of the theorem. \square

Now we are ready to prove the main result of this paper.

Theorem 8.11. *The BKL Conjecture 8.1 holds for an arbitrary $0^m 1^n$ -sequence.*

Proof. We shall proceed by induction on n . The base case when $n = 0$ is Theorem 8.3 (with $k = 0$), which is a Fock space reformulation of the classical Kazhdan-Lusztig conjecture. By induction hypothesis, for a given n , the BKL conjecture holds for all $0^m 1^n$ -sequences and for every m . By Theorem 8.10, the BKL conjecture holds for one particular $0^m 1^{n+1}$ -sequence. Now by Theorem 8.8 and Remark 8.9, the BKL conjecture holds for all $0^m 1^{n+1}$ -sequences. The induction is completed. \square

Remark 8.12. It follows from Theorem 8.11 that all the parabolic versions with even standard Levi subalgebras of the BKL conjecture hold, via similar comparisons as formulated in §4.3 and §8.5. Note however that our proof of Theorem 8.11 uses in an essential way a distinguished parabolic case which was established earlier in [CL] via the approach of super duality [CWZ, CW1].

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